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Double coset algebras

Robert May

Department of Mathematics and Computer Science, Longwood University, 201 High Street, Farmville, VA 23909, United States

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ABSTRACT

For a finite monoid M with unit e and an indexed family $\mathbf{G} = \{G_i\colon i \in I\}$ of subgroups of the group $G(M)$ of invertible elements in M, the complex vector space $A(M, \mathbf{G})$ with basis the double cosets of the form $G_i m G_j$, $m \in M$, has a natural multiplication yielding an associative $\mathbb{C}$-algebra which we call a double coset algebra. We construct two $\mathbb{Z}$-algebras with identity, $LGS(M, \mathbf{G})$ and $RGS(M, \mathbf{G})$, called the left and right generalized Schur algebras, which as $\mathbb{Z}$-modules are free with basis the double cosets. When M is the symmetric group $\mathfrak{S}_r$ and $\mathbf{G}$ is the family of Young subgroups indexed by compositions of r with at most n parts, $A(M, \mathbf{G})$ is isomorphic to the usual Schur algebra $S(n, r)$ and $LGS(M, \mathbf{G})$ corresponds to its standard $\mathbb{Z}$-form.

The structure constants we derive for these algebras provide multiplication rules useful for further analysis of the generalized Schur algebras. Our main result is then the observation that $LGS(M, \mathbf{G})$ and $RGS(M, \mathbf{G})$ are always $\mathbb{Z}$-forms for $A(M, \mathbf{G})$, that is, $A(M, \mathbf{G}) \cong \mathbb{C} \otimes_{\mathbb{Z}} LGS(M, \mathbf{G}) \cong \mathbb{C} \otimes_{\mathbb{Z}} RGS(M, \mathbf{G})$.

The Iwahori–Hecke algebra $H(M, G)$ corresponding to a finite monoid M and subgroup G is isomorphic to the double coset algebra $A(M, \mathbf{G})$ where $\mathbf{G} = \{G\}$ consists of a single set. So, as a special case of our result, we obtain $\mathbb{Z}$-forms $LGS(M, \mathbf{G})$ and $RGS(M, \mathbf{G})$ for arbitrary Iwahori–Hecke algebras. The existence of a $\mathbb{Z}$-form for any such $H(M, G)$ appears to be a new result.

1. Introduction

For a finite monoid M with unit e and an indexed family $\mathbf{G} = \{G_i\colon i \in I\}$ of subgroups of the group $G(M)$ of invertible elements in M consider the complex vector space $A(M, \mathbf{G})$ with basis the double cosets of the form $G_i m G_j$, $m \in M$. There is a natural multiplication on $A(M, \mathbf{G})$ yielding an associative $\mathbb{C}$-algebra which we call a double coset algebra. We also construct two $\mathbb{Z}$-algebras with identity, $LGS(M, \mathbf{G})$ and $RGS(M, \mathbf{G})$, which we call left and right generalized Schur algebras. As $\mathbb{Z}$-modules, these are free with basis the double cosets. When M is the symmetric group $\mathfrak{S}_r$ or the full transformation semigroup τ_r and $\mathbf{G}$ is the family of Young subgroups indexed by compositions of r with at most n parts, $A(M, \mathbf{G})$ is isomorphic to the

E-mail address: rmay@longwood.edu.

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usual Schur algebra $S(n,r)$ or the generalized Schur algebra of [5] and [4]. Then $LGS(M,\mathbf{G})$ corresponds to the known $\mathbb{Z}$-form for the usual Schur algebra, while $RGS(M,\mathbf{G})$ corresponds to the $\mathbb{Z}$-form in [4] for a generalized Schur algebra.

We derive structure constants for these algebras, providing useful multiplication rules for the Schur algebras. Our main result is then the observation that $LGS(M,\mathbf{G})$ and $RGS(M,\mathbf{G})$ are always $\mathbb{Z}$-forms for $A(M,\mathbf{G})$, that is, $A(M,\mathbf{G}) \cong \mathbb{C} \otimes_{\mathbb{Z}} LGS(M,\mathbf{G}) \cong \mathbb{C} \otimes_{\mathbb{Z}} RGS(M,\mathbf{G})$.

The Iwahori–Hecke algebra $H(M,G)$ corresponding to a finite monoid M and subgroup G (see [2,6,7]) is isomorphic to the double coset algebra $A(M,\mathbf{G})$ where $\mathbf{G} = \{G\}$ consists of a single set. So, as a special case of our result, we obtain $\mathbb{Z}$-forms $LGS(M,\mathbf{G})$ and $RGS(M,\mathbf{G})$ for arbitrary Iwahori–Hecke algebras. When M is itself a group the two $\mathbb{Z}$-forms are isomorphic and agree with the well-known $\mathbb{Z}$-form for this case. For general finite monoids M, the two $\mathbb{Z}$-forms are in general distinct and the existence of a $\mathbb{Z}$-form for any such $H(M,G)$ appears to be a new result. We remark that the $\mathbb{Z}$-forms obtained by Solomon in [7] and Godelle in [2] for special cases when $A(M,\mathbf{G})$ is the q-rook algebra or a deformation of the monoid algebra corresponding to a Renner monoid differ from either of our $\mathbb{Z}$-forms.

2. The standard double coset algebra

Let M be a finite monoid with identity e and $\mathbf{G} = \{G_i\colon\ i \in I\}$ be a family of subgroups of the group $G(M)$ of invertible elements in M indexed by a finite set I. (We may have $G_i = G_j$ for distinct indices $i \neq j$.) For each $i,j \in I$ there is an equivalence relation ${}_i\sim_j$ on M, defined by $m\,{}_i\!\sim_j n \Leftrightarrow n = \sigma m \pi$ for some $\sigma \in G_i$, $\pi \in G_j$, for which the equivalence classes are the double cosets of the form $G_i m G_j$, $m \in M$. Let ${}_iM_j$ be the set of such double cosets. Let M_I be the disjoint union of all the sets ${}_iM_j$, $i,j \in I$. More precisely, $M_I = \{(D,i,j)\colon\ i,j \in I,\ D \in {}_iM_j\}$.

Let $A = \mathbb{C}[M]$ be the monoid algebra over the complex numbers $\mathbb{C}$ and let ${}^iA^j$ be the subspace of A which is invariant under left multiplication by G_i and right multiplication by G_j:

$$ {}^iA^j = \{x \in A\colon\ \forall \sigma \in G_i,\ \forall \pi \in G_j,\ \sigma x \pi = x\}. $$

For any double coset $D \in {}_iM_j$ define $X(D) = \sum_{m \in D} m \in A$. Evidently $X(D) \in {}^iA^j$ and a simple calculation shows that $\{X(D)\colon\ D \in {}_iM_j\}$ is a basis for the subspace ${}^iA^j \subseteq \mathbb{C}[M]$. (For example, see [2] for the special case of Iwahori–Hecke algebras.)

Notice that the subspaces ${}^iA^j$ for distinct indices may have nontrivial intersection or may even coincide. We take disjoint copies of these subspaces and form the direct sum:

$$ A(M,\mathbf{G}) \equiv A_I \equiv \bigoplus_{i,j \in I} {}^iA^j. $$

Then the vector space A_I has a basis $\{X(D)\colon\ (D,i,j) \in M_I\}$ indexed by M_I. More generally, let $r : M_I \to \mathbb{C}^*$ be a scaling function which assigns to each double coset $(D,i,j) \in M_I$ a nonzero value $r(D,i,j) \in \mathbb{C} - \{0\}$. Then the set $\{r(D,i,j)X(D)\colon\ (D,i,j) \in M_I\}$ forms a basis for A_I. We will write

$$ b_r(D,i,j) = r(D,i,j)X(D) $$

for these basis elements.

We wish to make A_I into an associative algebra with identity. The construction is a natural generalization of the special case of Iwahori–Hecke algebras as given in [2]. For $i \in I$ we have $G_i = G_i e G_i \in {}_iM_i$. Then define $e_i = \frac{X(G_i)}{o(G_i)} \in {}^iA^i \subseteq A$, where $o(G_i)$ is the order of the group G_i. It is easy to check that each e_i is an idempotent in A and that ${}^iA^j = e_i A e_j$. We want the idempotents e_i to be orthogonal in A_I which is equivalent to taking $x * y = 0$ when $x \in {}^iA^j$, $y \in {}^kA^l$ and $j \neq k$. For the case $j = k$, notice that