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On quantum GL(n)

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ABSTRACT

The quantum GL(n) of Faddeev, Reshetikhin, and Takhtajan, and that of Dipper and Donkin are realized geometrically by using double partial flag varieties. As a consequence, the difference of these two Hopf algebras is caused by a twist of a cocycle in the multiplication.

1. Introduction

Let GL(n) be the general linear group and $\mathfrak{gl}(n)$ its Lie algebra. In [2], a geometric realization of quantum $\mathfrak{gl}(n)$ (or rather the q-Schur algebra) is given by using double flag varieties. From such a construction arises the so-called modified quantum $\mathfrak{gl}(n)$ and its canonical basis. Such classes of algebras play important roles in higher representation theory [11].

It is natural to ask if the quantum GL(n) itself admits a geometric realization. Since quantum GL(n) is, in principle, dual to quantum $\mathfrak{gl}(n)$, one may expect to get an answer from the dual construction of [2]. However, the answer to such a question is subtle, because the group GL(n) admits several quantizations: one by Faddeev, Reshetikhin, and Takhtajan [7,8], one by Dipper and Donkin [6], one by Takeuchi [21], and one by Artin, Schelter, and Tate [1].

In this paper, we show that the dual construction of [2] together with the coproduct defined by Grojnowski [10] and Lusztig [16] is isomorphic to the quantum GL(n) of Dipper and Donkin. The quantum GL(n) of Faddeev, Reshetikhin, and Takhtajan is also obtained from this setting by twisting a cocycle on the multiplication. In the geometric realization of both quantizations, the comultiplication is the same. This shows that the two quantizations are isomorphic as coalgebras, which was proved by Du et al. [5] by a different method twenty years ago. A closer look at the geometric construction yields that the basis E^M in the quantum GL(n) is the same as the basis consisting of all characteristic functions of certain orbits up to a twist. One interesting fact in the geometric realization is that the quantum determinants in both quantizations get identified with a certain Young symmetrizer. This symmetrizer gives rise to the determinant representation of GL(n), and the parameter in the quantization process does not appear.

The geometric setting is suitable for investigating many topics on quantum GL(n) such as the quantum Howe GL(m)–GL(n) duality (see [23,27]) and the (dual) canonical basis of quantum GL(n) in [26] and [15, 29.5]. We hope to come back to these topics in the future.

2. Quantum GL(n), definitions

We refer to [20,22,12,13,18,19,24,25,3,4] for more details. We shall recall the definitions of quantum GL(n).

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2.1. Quantum group $\mathrm{GL}_v(n)$ of Faddeev, Reshetikhin, and Takhtajan

Let $\mathbb{C}$ be the field of complex numbers. We fix a non-zero element v in $\mathbb{C}$. Note that the parameter v is denoted by q in the literature, such as [7,6]. We reserve the letter q for the finite field $\mathbb{F}_q$ of q elements.

Following [7], the quantum matrix algebra $A_v(n)$ of Faddeev, Reshetikhin, and Takhtajan is defined to be the unital associative algebra over $\mathbb{C}$ generated by the symbols E_{st} for $1 \leq i, j \leq n$ and subject to the following defining relations:

$$E_{ik}E_{jl} = E_{jl}E_{ik}, \quad \forall i > j, k < l;$$
$$E_{ik}E_{jl} = E_{jl}E_{ik} + (v - v^{-1})E_{jk}E_{il}, \quad \forall i > j, k > l;$$
$$E_{ik}E_{il} = vE_{il}E_{ik}, \quad \forall k > l;$$
$$E_{ik}E_{jk} = vE_{jk}E_{ik}, \quad \forall i > j.$$

The algebra $A_v(n)$ admits a bialgebra structure whose comultiplication

$$\Delta : A_v(n) \to A_v(n) \otimes A_v(n)$$

is defined by

$$\Delta(E_{st}) = \sum_{k=1}^{n} E_{ik} \otimes E_{kj} \quad \forall 1 \leq i, j \leq n, \tag{1}$$

and the counit $\epsilon : A_v(n) \to \mathbb{C}$ is given by $\epsilon(E_{st}) = \delta_{st}$ for any $1 \leq i, j \leq n$. Let

$$\det_v = \sum_{\sigma \in S_n} (-v)^{-l(\sigma)} E_{1,\sigma(1)} \ldots E_{n,\sigma(n)}, \tag{2}$$

be the quantum determinant of $A_v(n)$, where S_n is the symmetric group of n letters and l the length function. It is well known that $\det_v$ is a central element in $A_v(n)$. The quantum group $\mathrm{GL}_v(n)$ of Faddeev, Reshetikhin, and Takhtajan is (the would-be group whose coordinate ring is) the algebra obtained by localizing $A_v(n)$ at $\det_v$, i.e.,

$$A_v(n) \otimes_{\mathbb{C}} \mathbb{C}[T]/ < T\det_v - 1 > .$$

The comultiplication Δ of $A_v(n)$ extends naturally to a comultiplication of $\mathrm{GL}_v(n)$, still denoted by Δ, of $\mathrm{GL}_v(n)$. In particular, $\Delta(T) = T \otimes T$. Since T is the inverse of $\det_v$, we have $T = \det_v^{-1}$.

Fix two integers i and j among the set $\{1, \ldots, n\}$, and consider the subalgebra of $A_v(n)$ generated by the generators $E_{k,l}$ for $k \neq i$ and $l \neq j$. The resulting algebra is isomorphic to $A_v(n-1)$. So its quantum determinant, denoted by $A(i, j)$, is well defined. The antipode S of $\mathrm{GL}_v(n)$ is defined by

$$S(E_{st}) = (-v)^{j-i} A(j, i)\det_v^{-1}, \quad \forall 1 \leq i, j \leq n. \tag{3}$$

The datum $(\mathrm{GL}_v(n), \Delta, \epsilon, S)$ is a non-commutative, non-cocommutative Hopf algebra.

2.2. Quantum group $\mathrm{GL}_v^{DD}(n)$ of Dipper and Donkin

Following [6], the quantum matrix algebra $B_v(n)$ is defined to be the unital associative algebra over $\mathbb{C}$ generated by the symbols c_{st}, for $1 \leq i, j \leq n$, and subject to the following relations:

$$c_{ik}c_{jl} = vc_{jl}c_{ik}, \quad \forall i > j, k \leq l,$$
$$c_{ik}c_{jl} = c_{jl}c_{ik} + (v - 1)c_{jk}c_{il}, \quad \forall i > j, k > l,$$
$$c_{ik}c_{il} = c_{il}c_{ik}, \quad \forall i, l, k.$$

The triple $(B_v(n), \Delta, \epsilon)$, where Δ and ϵ are defined in Section 2.1, is again a bialgebra. Following [6, 4.1.7], let

$$\det_v^{DD} = \sum_{\sigma \in S_n} (-v)^{-l(\sigma)} c_{\sigma(1),1} \cdots c_{\sigma(n),n}, \tag{4}$$

be the quantum determinant of $B_v(n)$. (Note that this definition is equivalent to the other definitions in [6] for v invertible.) The quantum group $\mathrm{GL}_v^{DD}(n)$ is (the group whose coordinate algebra is) the algebra obtained from $B_v(n)$ by localizing at $\det_v^{DD}$.

Similar to the definition of $A(i, j)$, we can define the element $A(i, j)^{DD}$. Then the antipode of $\mathrm{GL}_v^{DD}(n)$ is given by

$$S^{DD}(c_{st}) = (-1)^{i+j} A^{DD}(j, i)\det_v^{DD,-1}, \quad \forall 1 \leq i, j \leq n, \tag{5}$$

where $\det_v^{DD,-1}$ is the inverse of $\det_v^{DD}$.

The datum $(\mathrm{GL}_v^{DD}(n), \Delta, \epsilon, S^{DD})$ is a Hopf algebra, where ϵ is defined in the same way as that of $\mathrm{GL}_v(n)$.