Journal of Pure and Applied Algebra 218 (2014) 1771–1782

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Star-regularity and regular completions

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ARTICLE INFO

Article history:
Received 24 October 2013
Received in revised form 11 November 2013
Available online 20 February 2014
Communicated by J. Adámek

MSC:
18A35; 18C05; 18D20; 08B20; 08B30; 08B05

ABSTRACT

In this paper we establish a new characterisation of star-regular categories, using a property of internal reflexive graphs, which is suggested by a recent result due to O. Ngaha Ngaha and the first author. We show that this property is, in a suitable sense, invariant under regular completion of a category in the sense of A. Carboni and E.M. Vitale. Restricting to pointed categories, where star-regularity becomes normality in the sense of the second author, this reveals an unusual behaviour of the exactness property of normality (i.e. the property that regular epimorphisms are normal epimorphisms) compared to other closely related exactness properties studied in categorical algebra.

1. Introduction

The context of a *star-regular category* was introduced in [10] (see also [11,12]) for a unification of some parallel results in the study of exactness properties in the context of regular categories, and exactness properties in the context of *normal categories* [20], i.e. pointed regular categories where every regular epimorphism is a normal epimorphism.

A star-regular category is a regular category equipped with an *ideal* in the sense of C. Ehresmann [7] (i.e. a class of morphisms which is closed under both left and right composition with morphisms in the category), satisfying suitable conditions (we recall the precise definition in Section 2 below). In particular, when the ideal consists of the zero morphisms in a pointed regular category, star-regularity states that every regular epimorphism is a normal epimorphism. In the universal-algebraic language, this can be equivalently reformulated by saying that any congruence is "generated" by its 0-class. Thus, for pointed varieties this is precisely the well-known 0-*regularity*, which was first studied (under a different name) in [8].

From the syntactic characterisation of 0-regular varieties obtained in [8], the following property of such varieties can be easily deduced: the congruence generated by the 0-class of a reflexive homomorphic relation

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[1] Partially supported by South African National Research Foundation.

http://dx.doi.org/10.1016/j.jpaa.2014.02.003

always contains the reflexive relation as a subrelation. In [13] it was shown that a similar fact holds more generally in any star-regular category. In this paper we refine and further generalise this result — see Theorem 2.2 below. This leads to showing that star-regular categories $\mathbb{C}$ which appear as a regular completion (in the sense of [6,22]) of a category $\mathbb{P}$ with weak finite limits, can be characterised using a property of reflexive graphs in $\mathbb{P}$, expressed in terms of the ideal of $\mathbb{C}$ restricted to $\mathbb{P}$ (see Theorem 3.3 below). Finally, at the end of the paper, we explicitly consider the pointed case, where star-regularity becomes normality. We point out some differences between how the regular completion behaves with respect to normality, and how it behaves with respect to some of the other closely related exactness properties, as established in [14,21,9].

2. Internal reflexive graphs

As briefly recalled in the Introduction, an *ideal* in a category $\mathbb{C}$ is a class $\mathcal{N}$ of morphisms of $\mathbb{C}$ such that for any composite fgh of three morphisms

$$W \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z, \qquad fgh : W \to Z$$

if $g \in \mathcal{N}$ then $fgh \in \mathcal{N}$. An ideal can be equivalently seen as a subfunctor of the hom-functor, and hence also as an enrichment in the category Set_2 of *pairs of sets*, i.e. the category arising from the Grothendieck construction applied to the covariant powerset functor $P : \mathsf{Set} \to \mathsf{Cat}$ (which is equivalent to the category of monomorphisms in Set). In [10], objects of Set_2, which are pairs (A, B) where A is a set and B is a subset of A, were called *multi-pointed sets*, since we can think of the subset B as a *set of base points* of A, in analogy with *pointed sets* where there is always only one base point. Then, a pair $(\mathbb{C}, \mathcal{N})$ where $\mathbb{C}$ is a category and $\mathcal{N}$ is an ideal of $\mathbb{C}$ is called a *multi-pointed category*. Pairs of sets, in the above sense, occur naturally in homological treatment of algebraic topology which in turn motivates the study of multi-pointed categories, as it has been thoroughly clarified by the work of M. Grandis on a non-additive generalisation of homological algebra [16,17]. In this theory, which was first elaborated in [15], multi-pointed categories enter as a natural generalisation of pointed categories. This aspect is still central in our work on "stars" [10–12], but, at the same time, we think of multi-pointed categories *also* as generalised ordinary (not necessarily pointed) categories. An ordinary category can be seen as a multi-pointed category where the ideal consists of all morphisms — we call this the *total context*. This gives only trivial examples in the realm of [17], while for what we do in a multi-pointed category, the total context is as important as the *pointed context* where every hom-set contains exactly one morphism from the ideal, in which case the ideal is uniquely determined and a multi-pointed category becomes a pointed category (see Remark 4.1 below).

Thinking of a multi-pointed category as a generalisation of a pointed category, in it one defines a kernel $k : K \to X$ of a morphism $f : X \to Y$ as a morphism such that fk belongs to the given ideal $\mathcal{N}$, and universal with this property: if fk' also belongs to $\mathcal{N}$ for some morphism $k' : K' \to X$, then $k' = ku$ for a unique morphism $u : K' \to K$, as illustrated in the following display:

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{f} & Y \\ {\scriptstyle \exists! u}\uparrow & \nearrow_{k'} & & & \\ K' & & & & \end{array}$$

To avoid confusion when we work with several ideals, we will say $\mathcal{N}$*-kernel* instead of *kernel*. This notion is one of the basic notions in the above-mentioned work of M. Grandis. In our work on multi-pointed categories, a central role is played by the following notion: given an ordered pair

$$X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \tag{1}$$