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Extendability of quadratic modules over a polynomial extension of an equicharacteristic regular local ring

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ABSTRACT

We prove that a quadratic $A[T]$-module Q with Witt index $(Q/TQ) \geqslant d$, where d is the dimension of the equicharacteristic regular local ring A, is extended from A. This improves a theorem of the second named author who showed it when A is the local ring at a smooth point of an affine variety over an infinite field. To establish our result, we need to establish a local–global principle (of Quillen) for the Dickson–Siegel–Eichler–Roy (DSER) elementary orthogonal transformations.

1. Introduction

Let A be a commutative Noetherian ring in which 2 is invertible and let B be the polynomial A-algebra $A[X_1, \ldots, X_n]$ in n indeterminates. Let $Q = (Q, q)$ be a quadratic space over B and let $Q_0 = (Q_0, q_0)$ be the reduction of Q modulo the ideal of B generated by $X_1, \ldots, X_n$. In [19], A.A. Suslin and V.I. Kopeĭko proved that if Q is stably extended from A and for every maximal ideal $\mathfrak{m}$ of A, the Witt index of $A_\mathfrak{m} \otimes_A (Q_0, q_0)$ is larger than the Krull dimension of A, then (Q, q) is extended from A. A shorter proof of this, due to I. Bertuccioni, can be found in [6] and another proof is in the thesis of the second named author.

In the thesis of the second named author (see [13,14]), it was shown that one can improve this result to Witt index $\geqslant d$, when A is a local ring at a non-singular point of an affine variety of dimension d over an infinite field. Moreover, a question was posed at the end of the thesis whether extendability can be shown for quadratic spaces with Witt index $\geqslant d$ over polynomial extensions of any equicharacteristic regular local ring of dimension d.

In this article, we establish this question affirmatively.

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A few words on the proof: The analysis of the equicharacteristic regular local ring is done by a patching argument, akin to the one developed by A. Roy in his paper [16]. This argument reduces the problem to the case of a complete equicharacteristic regular ring; which is a power series ring over a field, provided one can patch the information.

We found it useful to use A. Roy's elementary orthogonal transformations in [15] for quadratic spaces with a hyperbolic summand over a commutative ring. These transformations (over fields) are known as *Siegel* transformations or *Eichler* transformations in the literature: we give a brief historical statement of the development.

These transformations (in matrix form) of quadratic spaces (V, q) over finite fields first appeared on p. 126, p. 135 in L.E. Dickson's book "Linear groups: With an exposition of the Galois field theory" (1958), which is an unaltered republication of the first edition (Teubner, Leipzig, 1901). Later in "Sur les groupes classiques" (1948), J. Dieudonné extended these results over infinite fields.

These orthogonal transformations (in a matrix form) over general fields also appeared in a paper of C.L. Siegel: Über die analytische Theorie der quadratischen Formen II. Annals of Math. 36 (1935), 230–263.

Another interpretation occurs in his work "Über die Zetafunktionen indefiniter quadratischer Formen, II," Math. Zeitschrift, 1938, 398–426 (on p. 408). Here he used it to define the mass of representation of 0 by an indefinite quadratic form.

M. Eichler studied these transformations of $Q \perp H(k)$ in his study of the orthogonal group over fields k and made the first systematic use of them in his famous book "Quadratische Formen und orthogonale Gruppen", first published in 1952, and reprinted in 1974.

(Eichler credits Siegel's 1935 paper for introducing these transformations in the notes on §3 on p. 212 of his book, and also refers to the 1938 Zeitschrift paper of Siegel on p. 218. He does not seem to be aware of Dickson's work.)

A. Roy studied C.T.C. Wall's paper [21], who relied on Eichler's book. A. Roy rewrote the transformations of Eichler in Wall's paper. He then generalized these transformations in his thesis (1967) over any commutative ring R. We shall call these the DSER elementary orthogonal transformations or just (Roy's) elementary orthogonal transformation group.

Note that these transformations of Roy have been further extended to form rings by L.N. Vaserstein (when the ring is local), and A. Bak (to the general case) in their thesis (respectively).

We show that the patching process is possible by establishing a local–global principle for the elementary orthogonal group of a quadratic space with a hyperbolic summand. For this, we follow the broad outline of A.A. Suslin's method in [18] which led to a K_1 analogue of D. Quillen's local–global principle in [12]. Instead of using Suslin's 'theory of generic forms which are elementary', we follow the more 'hands on' approach via the yoga of commutators. For this, we have to first find an appropriate generating set for Roy's group; which is the primary objective of Section 2 (that this set generates the group is proved in Section 3, via V. Suresh's lemma in [17]). We record the commutator calculus in Section 4. These commutator calculations enable us to prove the local–global principle for Roy's group of orthogonal transformations over a polynomial extension.

As an interesting by-product, one realizes from the yoga of commutators in this elementary orthogonal group that it mimics G. Tang's well-known group in some features defined in [20], and the unitary group of Bass defined in [5]. The first named author intends to pursue the study of this group in more detail in a sequel article, where she hopes to establish A. Bak's type (see [4]) solvability theorem for the quotient group by the elementary subgroup.

Note: To make the reading effortless, we have placed the onerous (but straightforward) computations in this group in an article on the arXiv (see [1]) which can be accessed by any reader.

Finally, we have not attempted to study the 'Λ-ring variant' of this problem via the variant elementary orthogonal group as defined by A. Bak in his thesis (see [3]). We feel that it will throw more light on the interrelationship between all these groups; which will be carried out in a separate venture by the first named author.