Journal of Pure and Applied Algebra 218 (2014) 391–394

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

On Oguiso's $K3$ surface

Shingo Taki

School of Information Environment, Tokyo Denki University, 2-1200 Muzai Gakuendai, Inzai-shi, Chiba 270-1382, Japan

ARTICLE INFO

Article history:
Received 3 January 2013
Received in revised form 8 May 2013
Available online 22 June 2013
Communicated by S. Kovacs

MSC:
Primary: 14J28
Secondary: 14J50

ABSTRACT

It is known that a non-symplectic automorphism of order 32 on a $K3$ surface does not act trivially on the Néron–Severi lattice. Oguiso gave an example of a $K3$ surface with a non-symplectic automorphism of order 32 which acts on the Néron–Severi lattice as an involution. In this paper, we prove that a pair of a $K3$ surface and its non-symplectic automorphism of order 32 is isomorphic to Oguiso's example.

1. Introduction

This paper is devoted to a study of $K3$ surfaces with a non-symplectic automorphism. Let X be a $K3$ surface. In the following, we denote by S_X, T_X and ω_X the Néron–Severi lattice, the transcendental lattice and a nowhere vanishing holomorphic 2-form on X, respectively. Let σ be an automorphism on X of order I. It is called *non-symplectic* if and only if it satisfies $\sigma^*\omega_X = \zeta_I\omega_X$ where ζ_I is a primitive I-th root of unity.

The study of non-symplectic automorphisms of $K3$ surfaces was pioneered by V.V. Nikulin. He [6] proved that $\Phi(I)$ divides rank T_X, where Φ is the Euler function. In particular $\Phi(I) \leq \operatorname{rank} T_X$ and hence $I \leq 66$. It is easy to see that if I is prime-power then $I = 2^\alpha$ $(1 \leq \alpha \leq 5)$, 3^β $(1 \leq \beta \leq 3)$ or 5^γ $(1 \leq \gamma \leq 2)$.

Oguiso and Zhang [9] or Kondo [3] and Machida and Oguiso [4] have proved that the $K3$ surface with non-symplectic automorphisms of order 3^3 or 5^2, respectively, is unique up to isomorphism. Moreover these automorphisms act trivially on S_X. On the other hand, Vorontsov [13] announced that a non-symplectic automorphism of order 2^5 does not act trivially on S_X. (The claim was proved by Kondo [3].) Oguiso [8] gave an example of a $K3$ surface with a non-symplectic automorphism of order 32 which acts on S_X as an involution.

Example 1.1 *([8, Proposition 2(1)]).* Put

$$X_{\mathrm{og}} : y^2 = x^3 + t^2x + t^{11}, \qquad \sigma_{\mathrm{og}}(x, y, t) = (\zeta_{32}^{18}x, \zeta_{32}^{11}y, \zeta_{32}^{2}t).$$

Then X_{og} is a $K3$ surface and σ_{og} is a non-symplectic automorphism of order 32 which acts on $S_{X_{\mathrm{og}}}$ as an involution.

We call the pair $(X_{\mathrm{og}}, \langle\sigma_{\mathrm{og}}\rangle)$ *Oguiso's $K3$ surface* where $\langle\sigma_{\mathrm{og}}\rangle$ is the cyclic automorphism group generated by σ_{og}. The main purpose of this paper is to prove the following theorem.

Theorem 1.2. *Let X be a $K3$ surface and σ a non-symplectic automorphism of order* 32 *on X.*

(1) *The fixed locus of σ has exactly six points.*
(2) *A pair $(X, \langle\sigma\rangle)$ is isomorphic to Oguiso's $K3$ surface. Hence such a pair is unique up to isomorphism.*

Recently, Taki [12] studied non-symplectic automorphisms acting trivially on S_X, namely of prime-power order except 2^5.

E-mail address: staki@mail.dendai.ac.jp.

0022-4049/$ – see front matter
http://dx.doi.org/10.1016/j.jpaa.2013.06.009

Proposition 1.3. (1) *If X has a non-symplectic automorphism φ of order 4 acting trivially on S_X then $S_X = U, U(2), U \oplus D_4, U(2) \oplus D_4, U \oplus E_8, U \oplus D_8, U \oplus D_4^{\oplus 2}, U(2) \oplus D_4^{\oplus 2}, U \oplus E_8 \oplus D_4, U \oplus D_8 \oplus D_4$ or $U \oplus E_8^{\oplus 2}$. Moreover the fixed locus X^φ has the form*

$$X^\varphi = \begin{cases} \{P_1, P_2, P_3, P_4\} & \text{if } \operatorname{rank} S_X = 2, \\ \{P_1, P_2, \ldots, P_6\} \amalg \mathbb{P}^1 & \text{if } \operatorname{rank} S_X = 6, \\ \{P_1, P_2, \ldots, P_8\} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } \operatorname{rank} S_X = 10, \\ \{P_1, P_2, \ldots, P_{10}\} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } \operatorname{rank} S_X = 14, \\ \{P_1, P_2, \ldots, P_{12}\} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } \operatorname{rank} S_X = 18. \end{cases}$$

(2) *If X has a non-symplectic automorphism φ of order 8 acting trivially on S_X then $S_X = U \oplus D_4, U(2) \oplus D_4$ or $U \oplus D_4 \oplus E_8$. Moreover the fixed locus X^φ has the form*

$$X^\varphi = \begin{cases} \{P_1, P_2, \ldots, P_6\} \amalg \mathbb{P}^1 & \text{if } \operatorname{rank} S_X = 6, \\ \{P_1, P_2, \ldots, P_{12}\} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } \operatorname{rank} S_X = 14. \end{cases}$$

(3) *If X has a non-symplectic automorphism φ of order 16 acting trivially on S_X then $S_X = U \oplus D_4$ or $U \oplus D_4 \oplus E_8$. Moreover the fixed locus X^φ has the form*

$$X^\varphi = \begin{cases} \{P_1, P_2, \ldots, P_6\} \amalg \mathbb{P}^1 & \text{if } S_X = U \oplus D_4, \\ \{P_1, P_2, \ldots, P_{12}\} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } S_X = U \oplus D_4 \oplus E_8. \end{cases}$$

Here P_i is an isolated fixed point.

We summarize the contents of this paper. In Section 2, we study the action on the Néron–Severi lattice and the fixed locus of the automorphism by using the Lefschetz formulas. Then we prove Theorem 1.2(1). In Section 3, we study elliptic fibrations $\pi : X \to \mathbb{P}^1$ such that σ preserves each fiber of π. Section 4 gives the Proof of Theorem 1.2(2).

Throughout this article we shall denote by A_m, D_n, E_l the negative-definite root lattice of type A_m, D_n, E_l respectively. We denote by U the even indefinite unimodular lattice of rank 2.

2. Néron–Severi lattice and the fixed locus

Let X be a $K3$ surface and σ a non-symplectic automorphism of order 32 on X. We study the fixed locus of σ and prove that σ^* acts on S_X as an involution.

Lemma 2.1. *If $m = 1, 2, 4$ or 8 then the Euler characteristic of the fixed locus of σ^m is $2 + tr(\sigma^{*m}|S_X)$. In particular the trace of $\sigma^* \mid S_X$ is an integer.*

Proof. We apply the topological Lefschetz formula to the fixed locus X^{σ^m}: $\chi(X^{\sigma^m}) = \sum_{i=0}^4 (-1)^i \mathrm{tr}(\sigma^{*m}|H^i(X, \mathbb{R}))$. Since $\mathrm{tr}(\sigma^{*m}|T_X) = 0$ by [6, Theorem 3.1], we have $\chi(X^{\sigma^m}) = 1 - 0 + \mathrm{tr}(\sigma^{*m}|S_X) + \mathrm{tr}(\sigma^{*m}|T_X) - 0 + 1 = 2 + \mathrm{tr}(\sigma^{*m}|S_X)$. □

Note that $\sigma^* \mid S_X$ has an eigenvalue 1 corresponding to the pull back of an ample class of X/σ and rank $S_X = 6$ by [6, Theorem 3.1]. Assume that $\sigma^* \mid S_X \otimes \mathbb{C}$ is diagonalized as $\mathrm{diag}(1, \pm 1, \zeta_{32}^a, \zeta_{32}^{16+a}, \zeta_{32}^b, \zeta_{32}^{16+b})$. We remark that $\zeta_{32}^\alpha + \zeta_{32}^{-\alpha}$ is not an integer in general.

Lemma 2.2. *The integer a is divisible by 4. In particular σ^{*8} acts trivially on S_X.*

Proof. We remark that $\sigma^{*2} \mid S_X \otimes \mathbb{C} = \mathrm{diag}(1, 1, \zeta_{16}^a, \zeta_{16}^a, \zeta_{16}^b, \zeta_{16}^b)$. Since its trace is an integer, $b = 8 + a$. Thus $\sigma^{*4} \mid S_X \otimes \mathbb{C} = \mathrm{diag}(1, 1, \zeta_8^a, \zeta_8^a, \zeta_8^a, \zeta_8^a)$. By the same reason, $\zeta_8^a = \pm 1$. □

If the fixed locus X^σ contains a non-singular curve then X^{σ^8} also contains it. Thus X^σ contains at most one non-singular rational curve by Proposition 1.3(1). Hence X^σ is of the form $X^\sigma = \{P_1, P_2, \ldots, P_M\}$ or $\{P_1, P_2, \ldots, P_M\} \amalg \mathbb{P}^1$ where $M \leq 6$.

Proposition 2.3. *The fixed locus of σ has exactly six points.*

Proof. Let $P^{i,j}$ be an isolated fixed point given by the local action $\begin{pmatrix} \zeta_{32}^i & 0 \\ 0 & \zeta_{32}^j \end{pmatrix}$ and $m_{i,j}$ the number of isolated fixed points of type $P^{i,j}$. Hence $M = \sum_{i+j=33} m_{i,j}$.

Assume $X^\sigma = \{P_1, P_2, \ldots, P_M\} \amalg \mathbb{P}^1$. We apply the holomorphic Lefschetz formula ([1, page 542] and [2, page 567]) to X^σ:

$$\sum_{k=0}^{2} \mathrm{tr}(\sigma^*|H^k(X, \mathcal{O}_X)) = \sum_{i+j=33}^{M} a(P^{i,j}) + b(\mathbb{P}^1).$$

Here $a(P^{i,j}) = 1/((1 - \zeta_{32}^i)(1 - \zeta_{32}^j))$ and $b(\mathbb{P}^1) = (1 + \zeta_{32})/(1 - \zeta_{32})^2$. But it is easy to see that the calculation of the formula induces a contradiction. Thus $X^\sigma = \{P_1, P_2, \ldots, P_M\}$.