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On abelian subalgebras and ideals of maximal dimension in supersolvable Lie algebras

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ABSTRACT

In this paper, the main objective is to compare the abelian subalgebras and ideals of maximal dimension for finite-dimensional supersolvable Lie algebras. We characterise the maximal abelian subalgebras of solvable Lie algebras and study solvable Lie algebras containing an abelian subalgebra of codimension 2. Finally, we prove that nilpotent Lie algebras with an abelian subalgebra of codimension 3 contain an abelian ideal with the same dimension, provided that the characteristic of the underlying field is not 2. Throughout the paper, we also give several examples to clarify some results.

1. Introduction

Nowadays, there exists an extensive body of research of Lie theory due to its own importance from a theoretical point of view and also due to its applications to other fields such as Engineering, Physics, and Applied Mathematics. However, some aspects of Lie algebras remain unknown. Indeed, the classification of nilpotent and solvable Lie algebras is still an open problem, although the classification of certain other types of Lie algebra (such as semi-simple and simple ones) were already obtained in 1890, at least over the complex field. In order to make progress on these and other problems, the need for studying different properties of Lie algebras arises. For example, conditions on the lattice of subalgebras of a Lie algebra often lead to information about the Lie algebra itself. Studying abelian subalgebras and ideals of a finite-dimensional Lie algebra constitutes the main goal of this paper.

Throughout, L will denote a finite-dimensional Lie algebra over a field F. The assumptions on F will be specified in each result. Algebra direct sums will be denoted by $\oplus$, whereas vector space direct sums will be denoted by $\dot{+}$. We consider the following invariants of L:

$$\alpha(L) = \max\{\dim(A) \mid A \text{ is an abelian subalgebra of } L\},$$
$$\beta(L) = \max\{\dim(B) \mid B \text{ is an abelian ideal of } L\}.$$

Both invariants are important for many reasons. For example, they are very useful for the study of Lie algebra contractions and degenerations. There is a large literature, in particular for low-dimensional Lie algebras; see [7,4,11,13,6], and the references given therein.

The first author to deal with the invariant $\alpha(\mathfrak{g})$ was Schur [12], who studied the abelian subalgebras of maximal dimension contained in the Lie algebra of $n \times n$ square matrices in 1905. Schur proved that *the maximum number of linearly independent commuting $n \times n$ matrices over an algebraically closed field is* $\left[\frac{n^2}{4}\right] + 1$, which is the maximal dimension of abelian ideals

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Table 1
The invariant α for simple Lie algebras.

$\mathfrak{s}$	$\dim(\mathfrak{s})$	$\alpha(\mathfrak{s})$
$A_n,\ n \geq 1$	$n(n+2)$	$\left[\left(\frac{n+1}{2}\right)^2\right]$
B_3	21	5
$B_n,\ n \geq 4$	$n(2n+1)$	$\frac{n(n-1)}{2}+1$
$C_n,\ n \geq 2$	$n(2n+1)$	$\frac{n(n+1)}{2}$
$D_n,\ n \geq 4$	$n(2n-1)$	$\frac{n(n-1)}{2}$
G_2	14	3
F_4	52	9
E_6	78	16
E_7	133	27
E_8	248	36

of Borel subalgebras in the general linear Lie algebra $\mathfrak{gl}(n)$ (where $[x]$ denotes the integer part of a real number x). Let us note that this result was obtained only over an algebraically closed field such as the complex number field. Almost forty years later, in 1944, Jacobson [8] gave a simpler proof of Schur's results, extending them from algebraically closed fields to arbitrary fields. This fact allowed several authors to gain insight into the abelian subalgebras of maximal dimension of many different types of Lie algebra.

More specifically, for semisimple Lie algebras $\mathfrak{s}$, the invariant $\alpha(\mathfrak{s})$ has been completely determined by Malcev [10]. Since there are no abelian ideals in $\mathfrak{s}$, we have $\beta(\mathfrak{s}) = 0$. The value of α for simple Lie algebras is reproduced in Table 1. In this paper, we will study several properties of these invariants and compare them for supersolvable, solvable. and nilpotent Lie algebras.

We shall call L *supersolvable* if there is a chain $0 = L_0 \subset L_1 \subset \cdots \subset L_{n-1} \subset L_n = L$, where L_i is an i-dimensional ideal of L. The ideals $L^{(k)}$ of the *derived series* are defined by $L^{(0)} = L, L^{(k+1)} = [L^{(k)}, L^{(k)}]$ for $k \geq 0$; we also write L^2 for $L^{(1)}$ and L^3 for $[L^2, L]$. It is well known that every supersolvable Lie algebra is also solvable. Moreover, these classes coincide over an algebraically closed field of characteristic zero (Lie's theorem). There are, however, examples of solvable Lie algebras over algebraically closed fields of non-zero characteristic which are not supersolvable (see for instance [9, page 53] or [2]). The *Frattini ideal* of L, $\phi(L)$, is the largest ideal of L contained in all maximal subalgebras of L. We will denote the *centre* of L by $Z(L) = \{x \in L : [x, y] = 0, \forall y \in L\}$, and the *centraliser* of a subalgebra A of L by $C_L(A) = \{x \in L : [x, A] = 0\}$. Given a subalgebra A of L, the *core* of A, denoted by A_L, is the largest ideal of L contained in A. The *abelian socle* of L, Asoc L, is the sum of the minimal abelian ideals of L.

The structure of this paper is as follows. In Section 2, we give some bounds for the invariants α and β. In Section 3, we consider the classes of supersolvable, solvable, and nilpotent Lie algebras L with $\alpha(L) = n-1$ or $n-2$. In particular, we characterise n-dimensional solvable Lie algebras L for which $\alpha(L) = n-2$, and prove that every supersolvable Lie algebra L of dimension n with $\alpha(L) = n-2$ also satisfies $\beta(L) = n-2$. In Section 4, we show that the α and β invariants also coincide for nilpotent Lie algebras L with $\alpha(L) = n-3$, provided that F has characteristic different from 2. We also give an example to show that the restriction on F is necessary.

2. Some bounds on $\alpha(L)$ and $\beta(L)$

We shall call an L *metabelian* if L^2 is abelian. First, we have a bound on $\beta(L)$ for certain metabelian Lie algebras.

Proposition 2.1. *Let L be a metabelian Lie algebra of dimension n, and suppose that* $\dim L^2 = k$. *Then* $\dim(L/C_L(L^2)) \leq [k^2/4] + 1$. *If, further, L splits over L^2, then $\beta(L) \geq n - [k^2/4] - 1$.*

Proof. Let ad $: L \to \operatorname{Der} L^2$ be defined by ad $x(y) = [y, x]$ for all $y \in L^2$. Then ad is a homomorphism with kernel $C_L(L^2)$. It follows that $L/C_L(L^2) \cong D$, where D is an abelian subalgebra of Der $L^2 \cong gl(n, F)$. It follows from Schur's theorem on commuting matrices (see [8]) that $\dim(L/C_L(L^2) \leq [k^2/4] + 1$.

Now suppose that $L = L^2 \oplus B$, where B is an abelian subalgebra of L. Then $C_L(L^2) = L^2 \oplus B \cap C_L(L^2)$, which is an abelian ideal of L. □

We call L *completely solvable* if L^2 is nilpotent. Over a field of characteristic zero, every solvable Lie algebra is completely solvable. Next we note that, if L is completely solvable, has an abelian nilradical (so is metabelian), and the underlying field is perfect, then $\alpha(L)$ and $\beta(L)$ are easily identified. If $\bar{F}$ is the algebraic closure of F, we put $\bar{S} = S \otimes_F \bar{F}$ for every subalgebra S of L.

Lemma 2.2. $\alpha(\bar{L}) \geq \alpha(L)$, $\beta(\bar{L}) \geq \beta(L)$.

Lemma 2.3. *Let L be any solvable Lie algebra with nilradical N. Then $C_L(N) \subseteq N$.*