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Second maximum sum of element orders on finite groups

S.M. Jafarian Amiri *, Mohsen Amiri

Department of Mathematics, Faculty of Sciences, University of Zanjan, P.O. Box 45195-313, Zanjan, Iran

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ABSTRACT

Amiri, Jafarian Amiri and Isaacs proved that the cyclic group has maximum sum of element orders on all groups of the same order. In this article we characterize finite groups which have maximum sum of element orders among all noncyclic groups of the same order. This result confirms the conjecture posed in Jafarian Amiri (2013) [2].

1. Introduction and results

Let G be a finite group. If X is a nonempty subset of G, then we define $\psi(X) = \sum_{x\in X} o(x)$ where $o(x)$ is the order of x in G. In [1], the authors showed that if G is a noncyclic group of order n, then $\psi(G) < \psi(C_n)$ where C_n is the cyclic group of order n. In fact C_n has maximum sum of element orders among all groups of order n. It is a natural question that which groups have maximum sum of element orders on noncyclic groups of the same order. This question was answered among all noncyclic nilpotent groups in [2] as follows.

Theorem 1.1. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be a positive integer where $p_1 < p_2 < \cdots < p_k$ are primes and set $s = \min\{1, \ldots, k\}$ such that $\alpha_s > 1$. Suppose that G is a noncyclic nilpotent group of order n.*

(i) *If $p_1^{\alpha_1} \neq 8$, then $\psi(G) \leq \psi(C_{\frac{n}{p_s}} \times C_{p_s})$.*

(ii) *If $p_1^{\alpha_1} = 8$, then $\psi(G) \leq \psi(Q_8 \times C_{\frac{n}{8}})$*

where Q_8 is the quaternion group of order 8.

In this article we generalize the above theorem on all groups with specific orders as follows:

Theorem 1.2. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_t^{\alpha_t}$ be a positive integer where $p_1 < p_2 < \cdots < p_t$ are primes and $\alpha_1 > 1$. Suppose that G is a noncyclic group of order n. The following then hold.*

(a) *Suppose that $\alpha_1 \leq 3$ whenever $p_2^{\alpha_2} = 3$. Then*

(1) *If $p_1^{\alpha_1} \neq 8$, then $\psi(G) \leq \psi(C_{\frac{n}{p_1}} \times C_{p_1})$.*

(2) *If $p_1^{\alpha_1} = 8$, then $\psi(G) \leq \psi(Q_8 \times C_{\frac{n}{8}})$.*

Moreover if G is not nilpotent, then we have not the equalities.

(b) *Suppose that $p_2^{\alpha_2} = 3$ and $\alpha_1 > 3$. Then $\psi(G) \leq \psi(C_m \times F)$ where $m = p_3^{\alpha_3}\ldots p_t^{\alpha_t}$ and $F = C_3 \rtimes C_{2^{\alpha_1}}$ is a nonabelian group. Moreover if G is nilpotent, then we have not the equality.*

* Corresponding author.
E-mail addresses: sm_jafarian@znu.ac.ir (S.M. Jafarian Amiri), m.amiri77@gmail.com (M. Amiri).

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Table 1
Second maximum of ψ for $\alpha_1 = 1$.

n	Structure of group with second maximum of ψ	IsNilpotent
150	$C_{25} \times S_3$	No
605	$C_{55} \times C_{11}$	Yes
3 549	$C_{169} \times (C_7 : C_3)$	No
14 415	$C_5 \times (C_{961} : C_3)$	No

According to the hypotheses, the first part of Theorem 1.2 says that the second maximum of ψ on all groups of the same order occurs in the nilpotent groups and the part (b) says that the second maximum of ψ among all groups of the same order occurs in a nonnilpotent group. In Theorem 1.2, if we have $\alpha_1 = 1$, then it seems that the structure of groups having maximum sum of element orders among all noncyclic groups of the same order is complicated. For example see Table 1.

Also Theorem 1.2 confirms the conjecture posed in [2]. By Theorem 1.2 we have the following corollary:

Corollary 1.3. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ be a positive integer where $p_1 < p_2 < \cdots < p_t$ are primes and $\alpha_1 > 1$ and also assume that $\alpha_1 \le 3$ whenever $p_2^{\alpha_2} = 3$. Suppose that G is a noncyclic group of order n.*

(1) *If $p_1^{\alpha_1} \neq 8$, then $\frac{p_1^{2\alpha_1+1}+1}{p_1^{2\alpha_1-1}+p_1^3-p_1^2+1} \le \frac{\psi(C_n)}{\psi(G)}$.*
(2) *If $p_1^{\alpha_1} = 8$, then $\frac{27}{23} \le \frac{\psi(C_n)}{\psi(G)}$.*

We shall show that these bounds are sharp. Also this corollary improves the main result in [1].

Throughout this article all groups mentioned are assumed to be finite and most of our notation is standard and can be for instance be found in [3]. We denote $G^m = \langle \{x^m \mid x \in G\} \rangle$ for each positive integer m. Also φ is Euler's function. Let $p > 2$ be a prime. Then a p-group G is said to be powerful if $G' \le G^p$. Moreover by Theorem 3.1 in [4], we have $o(xy) \le \max\{o(x), o(y)\}$ for all x and y belong to a powerful group G.

2. Proof of the main result

We begin with the following lemmas which will be used later.

Lemma 2.1. *Let A and B be groups. Then $\psi(A \times B) \le \psi(A)\psi(B)$ and $\psi(A \times B) = \psi(A)\psi(B)$ if and only if $\gcd(|A|, |B|) = 1$.*

Proof. The proof is straightforward. □

Lemma 2.2 (*See Lemma C in [1]*). *Let p be the largest prime divisor of an integer $n > 1$. Then $\varphi(n) \ge \frac{n}{p}$.*

Lemma 2.3 (*See 5.14 in [3]*). *Let $P \in Syl_p(G)$, where G is a finite group and p is the smallest prime divisor of $|G|$, and P is cyclic. Then G has a normal p-complement.*

Lemma 2.4. *Let G be a finite group and $P \in Syl_p(G)$, where p is the largest prime divisor of $|G|$ and $|G|$ has at least two prime divisors. If G has an element of order $\frac{|G|}{d}$, where $d \le p$, then $G = P \rtimes K$ for some subgroup K of G and P is a powerful group. If in addition $d < p$, then P is cyclic and if $d = p$, then K is cyclic.*

Proof. Assume that $o(a) = \frac{|G|}{d}$ for some $a \in G$ and $|P| = p^\beta$ (we may consider $\beta \ge 2$). Then P has an element y of order $p^{\beta-1}$ and so $|\Phi(P)| = |P'P^p| \ge p^{\beta-2}$. Therefore P is a powerful p-group.

Suppose that $d < p$. Then $|G : \langle a \rangle| < p$ and so $\langle a \rangle$ contains a Sylow p-subgroup P of G. Therefore P is cyclic. Also we have $\langle a \rangle \subseteq N_G(P)$ and thus $|G : N_G(P)| < p$. This implies that $P \lhd G$ and so G has a subgroup K such that $G \cong P \rtimes K$.

Now suppose that $d = p$ and $|G| = n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ where $p_1 < p_2 < \cdots < p_t$ are primes and $\alpha_i \ge 1$. Then $p_t = p$. Since $o(a) = \frac{n}{p}$, there is an element $x \in P - \langle a \rangle$. Then $|\langle x \rangle \cap \langle a \rangle| = p^m \le \frac{o(x)}{p}$ which implies that $G = \langle x \rangle \langle a \rangle$. Therefore each Sylow p_i-subgroup of G is cyclic for $i = 1, 2, \dots, t-1$. It follows from the previous lemma that $G = P_1 K_1$ where $P_1 \in Syl_{p_1}(G)$, $K_1 \lhd G$ and $K_1 \cap P_1 = 1$. Again applying this lemma a second time, we see that $K_1 = P_2 K_2$ where $P_2 \in Syl_{p_2}(K_1)$, $K_2 \lhd K_1$ and $K_2 \cap P_2 = 1$. Continuing like this, we deduce that $P \lhd G$. Put $b = a^{|P|}$. Then we obtain $G = P \rtimes \langle b \rangle$. □

Lemma 2.5. (1) *Suppose that P is a powerful p-group and $G = P \rtimes K$, where $\gcd(p, |K|) = 1$. Then for all $y \in K$ and $x \in P$ we have $o(y) \mid o(xy)$ and $o(xy) \mid o(x)o(y)$.*
(2) *Let $G = K \rtimes H$, where K is an abelian subgroup of G. Then for all $y \in K$ and $x \in H$ we have $o(y) \mid o(xy)$ and $o(xy) \mid o(x)o(y)$.*

Proof. (1) Let $x \in P$ and $y \in K$. Since $P \lhd G$, there exists $x' \in P$ such that $yxy^{-1} = x'$ and so $o(x) = o(x')$. Therefore we have $(xy)(xy) = xx'y^2$ and also $o(xx') \mid o(x)$ because P is a powerful p-group. Thus $(xy)^k = x''y^k$ for each positive integer k such that $x'' \in P$ and $o(x'') \mid o(x)$. If $o(y) = m$, then $(xy)^{o(y)} = x''$ and so $o(xy) \mid o(x)o(y)$. Since $P \cap K = 1$, we have $o(y) \mid o(xy)$, as wanted.
(2) It is proved similarly. □