Journal of Pure and Applied Algebra 218 (2014) 583–601

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Support varieties for transporter category algebras

Fei Xu

Department of Mathematics, Shantou University, 243 University Road, Shantou, Guangdong 515063, China

ARTICLE INFO

Article history:
Received 12 November 2012
Received in revised form 30 May 2013
Available online 1 August 2013
Communicated by D. Nakano

MSC:
Primary: 16E40; 16P40; 20C05; 20J06; 55N25
Secondary: 57S17

ABSTRACT

Let G be a finite group. Over any finite G-poset $\mathcal{P}$ we may define a transporter category as the corresponding Grothendieck construction. The classifying space of the transporter category is the Borel construction on the G-space $B\mathcal{P}$, while the k-category algebra of the transporter category is the (Gorenstein) skew group algebra on the G-algebra $k\mathcal{P}$.

We introduce a support variety theory for the category algebras of transporter categories. It extends Carlson's support variety theory on group cohomology rings to equivariant cohomology rings. In the mean time it provides a class of (usually non selfinjective) algebras to which Snashall–Solberg's (Hochschild) support variety theory applies. Various properties will be developed. Particularly we establish a Quillen stratification for modules.

1. Introduction

Let G be a finite group and $\mathcal{P}$ a finite G-poset. Throughout this paper, we assume that k is an algebraically closed field of characteristic p, dividing the order of G. We are interested in a finite category $G \propto \mathcal{P}$, which is the Grothendieck construction on the G-poset $\mathcal{P}$ and which we will call a *transporter category* in this paper. When $G = \{e\}$ is trivial, $\{e\} \propto \mathcal{P} \cong \mathcal{P}$ and when $\mathcal{P} = \bullet$ is trivial, $G \propto \bullet \cong G$. A transporter category $G \propto \mathcal{P}$ is the algebraic or categorical predecessor of the Borel construction $EG \times_G B\mathcal{P}$ on the finite G-CW-complex $B\mathcal{P}$, in the sense that $B(G \propto \mathcal{P}) \simeq EG \times_G B\mathcal{P}$. Our interests in transporter categories come from the fact that the equivariant cohomology ring $\mathrm{H}^*_G(B\mathcal{P}, k) = \mathrm{H}^*(EG \times_G B\mathcal{P}, k)$ is Noetherian. Through an algebraic construction of the equivariant cohomology ring, we may introduce in a natural way modules over this ring and hence extend Carlson's support variety theory for finite group algebras to one for finite transporter category algebras.

Let us recall some historical developments in support variety theory. Suppose that X is a compact G-space. Quillen [22,23] proved that $\mathrm{H}^*_G(X)$ is Noetherian. Following his notation, we put $\mathrm{H}_G(X)$ to be $\mathrm{H}^*_G(X)$ if $p = 2$ or $\mathrm{H}^{ev}_G(X)$, the even part of the equivariant cohomology ring, if $p \geq 3$. When $X = \bullet$ is just a point fixed by G, the equivariant ring reduces to the group cohomology ring and we shall write $\mathrm{H}^*_G = \mathrm{H}^*_G(\bullet)$ and $\mathrm{H}_G = \mathrm{H}_G(\bullet)$. Quillen's work began with the observation that the graded ring $\mathrm{H}_G(X)$ is commutative Noetherian. It enabled him to define a homogeneous affine variety $V_{G,X}$ as the maximal ideal spectrum $\mathrm{MaxSpec}\, \mathrm{H}_G(X)$, and subsequently described it in terms of $V_E = V_{E,\bullet} = \mathrm{MaxSpec}\, \mathrm{H}_E$, where E runs over the set of all elementary abelian p-subgroups of G such that $X^E \neq \emptyset$. This is what we nowadays refer to as the Quillen stratification. Restricting to the special case of $X = \bullet$, based on the fact that $\mathrm{Ext}^*_{kG}(M, M)$ is finitely generated over $\mathrm{H}^*_G \cong \mathrm{Ext}^*_{kG}(k, k)$, Carlson [11] extended Quillen's work by attaching to every finitely generated kG-module M a subvariety of $V_G = V_{G,\bullet}$, denoted by $V_G(M) = \mathrm{MaxSpec}\, \mathrm{H}_G/I_G(M)$, called the (cohomological) support variety of M, where $I_G(M)$ is the kernel of the following map

$$\phi_M = - \otimes_k M : \mathrm{H}^*_G \cong \mathrm{Ext}^*_{kG}(k, k) \to \mathrm{Ext}^*_{kG}(M, M).$$

Especially since ϕ_k is the identity, $V_G = V_G(k)$. Following Carlson's construction, Avrunin and Scott [5] quickly generalized the Quillen stratification from V_G to $V_G(M)$. By showing that the support varieties are well-behaved with respect to module

E-mail address: fxu@stu.edu.cn.

0022-4049/$ – see front matter
http://dx.doi.org/10.1016/j.jpaa.2013.07.006

operations, gradually Benson, Carlson and many others developed a remarkable theory, being a significant progress in group representations and cohomology. Since then, some other analogous support variety theories have been introduced for restricted Lie algebras [16], for finite group schemes [6,17], for complete intersections [4] and for certain finite-dimensional algebras [20,14,24].

Quillen's work on equivariant cohomology rings has not been fully exploited, partially because there existed no suitable modules which $H^*_G(X)$ (hence $H_G(X)$) acts on or maps to, as in Carlson's theory. In this article, we attempt to use category algebras to solve the problem: if $X = B\mathcal{P}$ comes from a finite G-poset, then we consider the category algebra $k(G \propto \mathcal{P})$ of the transporter category $G \propto \mathcal{P}$, based on which we will generalize Carlson's theory. In fact, let $\underline{k}$ be the trivial $k(G \propto \mathcal{P})$-module (see Section 2.2). Then $\mathrm{Ext}^*_{k(G\propto\mathcal{P})}(\underline{k}, \underline{k})$ is a graded commutative ring and there exists a natural ring isomorphism

$$\mathrm{Ext}^*_{k(G\propto\mathcal{P})}(\underline{k}, \underline{k}) \cong H^*(EG \times_G B\mathcal{P}, k) = H^*_G(B\mathcal{P}, k).$$

We shall call the above ring the *ordinary cohomology ring* of $k(G \propto \mathcal{P})$ (instead of the equivariant cohomology ring), as opposed to the *Hochschild cohomology ring* of $k(G \propto \mathcal{P})$. Then we define $V_{G\propto\mathcal{P}} = V_{G,B\mathcal{P}} = \mathrm{MaxSpec}\, H_G(B\mathcal{P})$. The virtue of having an entirely algebraic construction of the equivariant cohomology theory is that it allows us to consider

$$\mathrm{Ext}^*_{k(G\propto\mathcal{P})}(\mathfrak{M}, \mathfrak{N})$$

for any finitely generated $\mathfrak{M}, \mathfrak{N} \in k(G \propto \mathcal{P})$-mod, and moreover construct a map

$$\phi_{\mathfrak{M}} = -\hat{\otimes}\mathfrak{M} : \mathrm{Ext}^*_{k(G\propto\mathcal{P})}(\underline{k}, \underline{k}) \to \mathrm{Ext}^*_{k(G\propto\mathcal{P})}(\mathfrak{M}, \mathfrak{M}).$$

Here $\hat{\otimes}$ is the tensor product in the closed symmetric monoidal category $(k(G \propto \mathcal{P})\text{-mod}, \hat{\otimes}, \underline{k})$. Note that $\underline{k}$ serves as the tensor identity. Since we have shown in [29] that $\mathrm{Ext}^*_{k(G\propto\mathcal{P})}(\mathfrak{M}, \mathfrak{N})$ is finitely generated over the ordinary cohomology ring, we may define the support variety of $\mathfrak{M} \in k(G \propto \mathcal{P})$-mod as $V_{G\propto\mathcal{P}}(\mathfrak{M}) = \mathrm{MaxSpec}\, H_G(B\mathcal{P})/I_{G\propto\mathcal{P}}(\mathfrak{M})$, where $I_{G\propto\mathcal{P}}(\mathfrak{M})$ is the kernel of $\phi_{\mathfrak{M}}$. Especially $V_{G\propto\mathcal{P}} = V_{G\propto\mathcal{P}}(\underline{k})$. When $\mathcal{P} = \bullet$, the is exactly Carlson's construction because $k(G \propto \bullet) \cong kG$, $\underline{k}$ becomes the trivial kG-module k and $\hat{\otimes}$ reduces to $\otimes_k$ under the circumstance.

As a surprising consequence of our investigations of transporter category algebras, we assert that Snashall–Solberg's (Hochschild) support variety theory (for Gorenstein algebras) applies to every block of a finite transporter category algebra. Furthermore, our support variety theory is closely related with Snashall–Solberg's as what happens in the case of group algebras and their blocks. A notable point is that the block algebras of a transporter category algebra are usually non-selfinjective and non-commutative, opposing to the cases of (selfinjective) Hopf algebras [11,6,17] and of commutative Gorenstein algebras [4] considered by others.

This paper is organized as follows. Section 2 recalls the definitions of the transporter category, the category algebra and the category cohomology. Various necessary constructions are recorded for the convenience of the reader. Here we show a transporter category algebra is Gorenstein and the ordinary cohomology ring of such an algebra is identified with an equivariant cohomology ring. Then in Section 3, we define the support variety for a module over a transporter category algebra. To motivate the reader, we describe the relevant works of Carlson, Linckelmann and Snashall–Solberg, before we develop some standard properties. Sections 4 and 5 contain various properties of support varieties, including the Quillen stratification for modules, as well as results related with sub-transporter categories and tensor products.

2. Preliminaries

In this section, we recall the definition of a transporter category and some background in category algebras. Throughout this article we will only consider finite categories, in the sense that they have finitely many morphisms. Thus a group G, or a G-poset $\mathcal{P}$, is always finite.

The morphisms in a poset are customarily denoted by $\leq$. The *dimension* of a poset $\mathcal{P}$, $\dim \mathcal{P}$, is defined to be the maximal integer n such that a chain of non-isomorphisms $x_0 < x_1 < \cdots < x_n$ exists in $\mathcal{P}$.

Any G-set is usually regarded as a G-poset with trivial relations. One the other hand, since in a G-poset $\mathcal{P}$, both $\mathrm{Ob}\,\mathcal{P}$ and $\mathrm{Mor}\,\mathcal{P}$ are naturally G-sets, we shall use terminologies for G-sets in our situation without further comments.

2.1. Transporter categories as Grothendieck constructions

We deem a group as a category with one object, usually denoted by $\bullet$. The identity of a group G is always named e. We say a poset $\mathcal{P}$ is a G-poset if there exists a functor F from G to $\mathfrak{sCats}$, the category of small categories, such that $F(\bullet) = \mathcal{P}$. It is equivalent to saying that we have a group homomorphism $G \to \mathrm{Aut}(\mathcal{P})$. The *Grothendieck construction* on F will be called a *transporter category*.

Definition 2.1.1. Let G be a group and $\mathcal{P}$ a G-poset. The transporter category $G \propto \mathcal{P}$ has the same objects as $\mathcal{P}$, that is, $\mathrm{Ob}(G \propto \mathcal{P}) = \mathrm{Ob}\,\mathcal{P}$. For $x, y \in \mathrm{Ob}(G \propto \mathcal{P})$, a morphism from x to y is a pair $(g, gx \leq y)$ for some $g \in G$.

If $(g, gx \leq y)$ and $(h, hy \leq z)$ are two morphisms in $G \propto \mathcal{P}$, then their composite is easily seen to be $(hg, (hg)x \leq z) = (h, hy \leq z) \circ (g, gx \leq y)$.