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Lifting via cocycle deformation

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ABSTRACT

We develop a strategy to compute all liftings of a Nichols algebra over a finite dimensional cosemisimple Hopf algebra. We produce them as cocycle deformations of the bosonization of these two. In parallel, we study the shape of any such lifting.

1. Introduction

Let A be a finite-dimensional Hopf algebra whose coradical is a Hopf subalgebra H. Then the graded algebra associated to the coradical filtration of A is again a Hopf algebra, which is given by a smash product $\operatorname{gr} A \simeq R\#H$, for $R = \bigoplus_{n\geq 0} R^n$ a graded Hopf algebra in ${}^H_H\mathcal{YD}$, the category of Yetter–Drinfeld modules over H. Let $V = R^1$, then the subalgebra of R generated by V is the *Nichols algebra* $\mathcal{B}(V)$ [8]; this is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ which is also defined for every $V \in {}^H_H\mathcal{YD}$ by a universal quotient $T(V)/\mathcal{J}(V)$, for $\mathcal{J}(V)$ an ideal generated by homogeneous elements of degree ≥ 2.

If $\operatorname{gr} A = \mathcal{B}(V)\#H$, then A is called a *lifting* or *deformation* of $\mathcal{B}(V)$ (over H). Hence, deformations of $\mathcal{B}(V)$ give rise to new examples of Hopf algebras. Moreover, there are classes of Hopf algebras (as pointed Hopf algebras over abelian groups) in which every example arises as such a deformation.

1.1. The problem

In this article, we develop a strategy to compute all the liftings or deformations of a Nichols algebra. More precisely, we consider

$$\text{a Hopf algebra } H \text{ which is finite-dimensional and cosemisimple;} \tag{1.1}$$

$$V \in {}^H_H\mathcal{YD} \text{ such that } \dim V < \infty \text{ and } \mathcal{J}(V) \text{ is finitely generated.} \tag{1.2}$$

The problem is to describe all Hopf algebras A such that

$$\operatorname{gr} A \simeq \mathcal{B}(V)\#H. \tag{1.3}$$

Notice that the coradical of A is isomorphic to H by (1.3), see [8]. This problem is one of the steps in the Lifting Method [7,8], see also the generalization proposed in [3]. To deal with it, we split it into two parts:

(a) To detect the shape of all possible deformations.
(b) To show that these proposed deformations actually are so.

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Problem (a) is usually taken by examination of the comodule structure of the first term of the coradical filtration, what would give possible deformations by defining relations, see Section 4.

However it is not apparent that the proposed deformations have the desired property; namely, such deformation A would bear an epimorphism $\mathcal{B}(V)\#H \twoheadrightarrow \operatorname{gr} A$ but whether this is an isomorphism requires an extra reasoning. This is Problem (b) and there have been different approaches to face up to it: the Diamond Lemma [7,5,10]; a reduction to the first term of the coradical filtration followed by some representation theory, assuming that the Nichols algebra is quadratic [22]; a combination of deformation by cocycles and an examination of the PBW basis [9].

We briefly recall this last approach highlighting some features that are present in the strategy below; see [9] for more details and undefined notation. There, H is assumed to be the group algebra of a finite abelian group Γ (with some restrictions on the order) and $V \in {}^H_H\mathcal{YD}$ has a finite-dimensional Nichols algebra; therefore, by the restrictions alluded to, V is of Cartan type and gives rise to a Dynkin diagram Δ. The defining ideal $\mathcal{J}(V)$ is generated by three kind of relations:

(i) Serre relations in the same connected component of Δ,
(ii) Serre relations between vertices in different connected components,
(iii) powers of root vectors.

It is then shown that in any deformation A the Serre relations in the same connected component still hold, and the other relations deform respectively to the so-called linking relations, controlled by a family of parameters $\boldsymbol{\lambda}$, and the so-called power of root vector relations, controlled by a second family of parameters $\boldsymbol{\mu}$. Hence the A should be of the form $u(\mathcal{D}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = T(V)\#H/\mathcal{J}$, where the ideal $\mathcal{J}$ is generated by:

(i) Serre relations (in the same connected component),
(ii) linking relations,
(iii) power of root vector relations.

To show that $u(\mathcal{D}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ has the desired dimension $\dim \mathcal{B}(V)|\Gamma|$, the procedure in [9] goes as follows.

(a) Let $U(\mathcal{D}, \boldsymbol{\lambda}) = T(V)\#H/\mathcal{J}_0$, where the ideal $\mathcal{J}_0$ is generated by the Serre relations (in the same connected component) and the linking relations. Then $U(\mathcal{D}, \boldsymbol{\lambda})$ has the "right" basis; it is proved by induction on the number of connected components, via cocycle deformation in the inductive step.
(b) Finally, $u(\mathcal{D}, \boldsymbol{\lambda}) = U(\mathcal{D}, \boldsymbol{\lambda})/\mathcal{J}_1$, where $\mathcal{J}_1$ is generated by the power of root vector relations, has the right dimension by a delicate argument using centrality of these last relations in $U(\mathcal{D}, \boldsymbol{\lambda})$.

1.2. The background

The family of Hopf algebras $u(\mathcal{D}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ contains the liftings of quantum linear spaces defined in [7]. It was shown in [33] that these liftings of quantum linear spaces are cocycle deformations of their associated graded Hopf algebras. Further work in this direction was done in [15,14,26]; in this last paper it was stated that any Hopf algebra $u(\mathcal{D}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is a cocycle deformation of its associated graded Hopf algebra, but the argument had a gap and a complete proof was given in [35].

The result in [35] is first extended to the non-abelian case in [24] where it is shown that every finite-dimensional pointed Hopf algebra H over $\mathbb{S}_3$ or $\mathbb{S}_4$ is again a cocycle deformation of $\operatorname{gr} H$. In [11] it is shown that this is also the case for finite-dimensional copointed Hopf algebras over $\mathbb{S}_3$. Also, in [25] this is shown for some pointed or copointed Hopf algebras associated to affine racks. In all of these papers the results are achieved by computing Hopf biGalois objects. In [23], the authors pick up the work in [26] to explicitly compute cocycles as exponentials of Hochschild 2-cocycles. They show that every finite-dimensional pointed Hopf algebra H over the dihedral groups D_{4t} is a cocycle deformation of $\operatorname{gr} H$.

1.3. The strategy

In the present paper, we propose to reverse the order and start by computing all cocycle deformations following ideas in [35]. Observe that, since a deformation by cocycle affects only the multiplication, the coradical filtration of a cocycle deformation A of $\mathcal{B}(V)\#H$ remains unchanged, hence it is isomorphic to $\mathcal{B}(V)\#H$ as coalgebras. Also, it is possible to decide when A is a lifting of $\mathcal{B}(V)$ over H.

Set $\mathcal{T}(V) = T(V)\#H$, $\mathcal{H} = \mathcal{B}(V)\#H$. Our strategy is as follows:

(a) We decompose a minimal set of generators of the ideal defining $\mathcal{B}(V)$ and recover $\mathcal{H}$ as the last link in a chain of subsequent quotients $\mathcal{T}(V) \twoheadrightarrow \mathcal{B}_1\#H \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{B}_n\#H \twoheadrightarrow \mathcal{H}$. We choose this decomposition in such a way that every intermediate quotient is achieved by dividing by primitive elements in $\mathcal{B}_i$, $i = 1, \dots, n$.
(b) At each step, we compute the Galois objects of $\mathcal{H}_{i+1}$ as quotients of the Galois objects of $\mathcal{H}_i$, following the results in [27]. We start with the trivial Galois object for $\mathcal{T}(V)$. In the final step, we have a set Λ of Galois objects of $\mathcal{H}$ and hence a list of cocycle deformations L, which arise as $L \simeq L(\mathcal{A}, \mathcal{H})$, for $\mathcal{A} \in \Lambda$ as in [40].
(c) We check that any lifting is obtained as one of these deformations.

The paper is organized as follows. In Section 2 we fix the notation and introduce the preliminaries on Hopf algebras, Nichols algebras, cocycles and Hopf Galois objects. In Section 3 we recall the two theorems in [27] about cleft and Galois