Journal of Pure and Applied Algebra 219 (2015) 3263–3278

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Generalised bialgebras and entwined monads and comonads

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A R T I C L E   I N F O

Article history:
Received 1 April 2014
Received in revised form 29 September 2014
Available online 13 November 2014
Communicated by C. Kassel

MSC:
18C20; 18D50; 16T10; 16T15

A B S T R A C T

J.-L. Loday has defined generalised bialgebras and proved structure theorems in this setting which can be seen as general forms of the Poincaré–Birkhoff–Witt and the Cartier–Milnor–Moore theorems. It was observed by the present authors that parts of the theory of generalised bialgebras are special cases of results on entwined monads and comonads and the corresponding mixed bimodules. In this article the Rigidity Theorem of J.-L. Loday is extended to this more general categorical framework.

1. Introduction

The introduction of *entwining structures* between an algebra and a coalgebra by T. Brzeziński and S. Majid in [2] opened new perspectives in the mathematical treatment of quantum principal bundles. It turned out that these structures are special cases of distributive laws treated in J. Beck's paper [1]. The latter were also used by D. Turi and G. Plotkin [16] in the context of operational semantics.

These observations led to a revival of the investigation of various forms of distributive laws. In a series of papers [13–15] it was shown how they allow for formulating the theory of Hopf algebras and Galois extensions in a general categorical setting.

On the other hand, *generalised bialgebras* as defined in J.-L. Loday [8, Section 2.1], are vector spaces which are algebras over an *operad* $\mathscr{A}$ and coalgebras over a *cooperad* $\mathscr{C}$. Moreover, the operad $\mathscr{A}$ and the cooperad $\mathscr{C}$ are required to be related by a distributive law. Since any operad $\mathscr{A}$ yields a monad $\mathcal{T}_{\mathscr{A}}$ and $\mathscr{A}$-algebras are nothing but $\mathcal{T}_{\mathscr{A}}$-modules, and similarly any cooperad $\mathscr{C}$ yields a comonad $\mathcal{G}_{\mathscr{C}}$ and $\mathscr{C}$-coalgebras are nothing but $\mathcal{G}_{\mathscr{C}}$-comodules, generalised bialgebras have interpretations in terms of bimodules over a bimonad in the sense of [14].

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http://dx.doi.org/10.1016/j.jpaa.2014.10.013
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The purpose of the present paper is to make this relationship more precise (as proposed in [14, 2.3]). On the one hand, given a monad $\mathcal{T}$ and a comonad $\mathcal{G}$ on a category $\mathbb{A}$ together with a mixed distributive law λ between them, we provide in Theorem 4.1 conditions for which a functor from $\mathbb{A}$ to the category of TG-bimodules in $\mathbb{A}$ is an equivalence of categories. In Theorem 5.8 we concentrate on the special case when the monad and the comonad share the same underlying functor. These are general theorems, which do not depend on the shape of the functors underlying the monads and comonads. On the other hand, when considering operads and cooperads the functors involved are analytic in the sense of A. Joyal [7] and, in particular, are graded and connected. In Section 6, we show that the general theory together with the grading lead to the Rigidity Theorem [8, 2.5.1] as a special case (without using idempotents as in [8]). In Corollary 6.9 we focus on the special case when the operad and cooperad share the same underlying analytic functor and prove (under mild conditions) that the category of generalised bialgebras is equivalent to the category of vector spaces.

2. Comodules and adjoint functors

In this section we provide basic notions and properties of comodule functors and adjoint pairs of functors. Throughout the paper $\mathbb{A}$ and $\mathbb{B}$ will denote any categories.

2.1. Monads and comonads. Recall that a *monad* $\mathcal{T}$ on $\mathbb{A}$ is a triple (T, m, e) where $T : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $m : TT \to T$, $e : 1 \to T$ satisfying associativity and unitality conditions. A $\mathcal{T}$-*module* is an object $a \in \mathbb{A}$ with a morphism $h : T(a) \to a$ subject to associativity and unitality conditions. The (Eilenberg–Moore) category of $\mathcal{T}$-modules is denoted by $\mathbb{A}_{\mathcal{T}}$ and there is a free functor

$$\phi_{\mathcal{T}} : \mathbb{A} \to \mathbb{A}_{\mathcal{T}}, \quad a \mapsto \big(T(a), m_a\big),$$

which is left adjoint to the forgetful functor

$$U_{\mathcal{T}} : \mathbb{A}_{\mathcal{T}} \to \mathbb{A}, \quad (a, h) \mapsto a.$$

Dually, a *comonad* $\mathcal{G}$ on $\mathbb{A}$ is a triple (G, δ, ε) where $G : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $\delta : G \to GG$, $\varepsilon : G \to 1$, and $\mathcal{G}$-*comodules* are objects $a \in \mathbb{A}$ with morphisms $\theta : a \to G(a)$. Both notions are subject to coassociativity and counitality conditions. The (Eilenberg–Moore) category of $\mathcal{G}$-comodules is denoted by $\mathbb{A}^{\mathcal{G}}$ and there is a cofree functor

$$\phi^{\mathcal{G}} : \mathbb{A} \to \mathbb{A}^{\mathcal{G}}, \quad a \mapsto \big(G(a), \delta_a\big),$$

which is right adjoint to the forgetful functor

$$U^{\mathcal{G}} : \mathbb{A}^{\mathcal{G}} \to \mathbb{A}, \quad (a, \theta) \mapsto a.$$

2.2. $\mathcal{G}$-comodule functors. For a comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$, a functor $F : \mathbb{B} \to \mathbb{A}$ is a *left $\mathcal{G}$-comodule* if there exists a natural transformation $\alpha_F : F \to GF$ inducing commutativity of the diagrams

$$\begin{tikzcd} F \arrow[r, "\alpha_F"] \arrow[dr, equal] & GF \arrow[d, "\varepsilon F"] \\ & F, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\alpha_F"] \arrow[d, "\alpha_F"'] & GF \arrow[d, "\delta F"] \\ GF \arrow[r, "G\alpha_F"'] & GGF. \end{tikzcd}$$

Symmetrically, one defines right $\mathcal{G}$-comodules.