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Algebras simple with respect to a Taft algebra action

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A R T I C L E I N F O

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A B S T R A C T

Algebras simple with respect to an action of a Taft algebra $H_{m^2}(\zeta)$ deliver an interesting example of H-module algebras that are H-simple but not necessarily semisimple. We describe finite dimensional $H_{m^2}(\zeta)$-simple algebras and prove the analog of Amitsur's conjecture for codimensions of their polynomial $H_{m^2}(\zeta)$-identities. In particular, we show that the Hopf PI-exponent of an $H_{m^2}(\zeta)$-simple algebra A over an algebraically closed field of characteristic 0 equals $\dim A$. The groups of automorphisms preserving the structure of an $H_{m^2}(\zeta)$-module algebra are studied as well.

The notion of an H-(co)module algebra is a natural generalization of the notion of a graded algebra, an algebra with an action of a group by automorphisms, and an algebra with an action of a Lie algebra by derivations. In particular, if $H_{m^2}(\zeta)$ is the m^2-dimensional Taft algebra, an $H_{m^2}(\zeta)$-module algebra is an algebra endowed both with an action of the cyclic group of order m and with a skew-derivation satisfying certain conditions. The Taft algebra $H_4(-1)$ is called Sweedler's algebra.

The theory of gradings on matrix algebras and simple Lie algebras is a well developed area [3,6]. Quaternion $H_4(-1)$-extensions and related crossed products were considered in [9]. In [14], the author classified all finite dimensional $H_4(-1)$-simple algebras. Here we classify finite dimensional $H_{m^2}(\zeta)$-simple algebras over an algebraically closed field (Sections 2–3). The classification requires essentially new ideas in comparison with [14] since the Jacobson radical of such an algebra A can have nonzero multiplication and we have to study series of graded (A, A)-bimodules in A with irreducible factors. The proof becomes more ring-theoretical. In addition, the formula for the multiplication in A involves quantum binomial coefficients. (See Section 3.)

Amitsur's conjecture on asymptotic behaviour of codimensions of ordinary polynomial identities was proved by A. Giambruno and M.V. Zaicev [10, Theorem 6.5.2] in 1999.

Suppose an algebra is endowed with a grading, an action of a group G by automorphisms and anti-automorphisms, an action of a Lie algebra by derivations or a structure of an H-module algebra for some

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Hopf algebra H. Then it is natural to consider, respectively, graded, G-, differential or H-identities [1,2,4,7,15].

The analog of Amitsur's conjecture for polynomial H-identities was proved under wide conditions by the author in [12,13]. However, in those results the H-invariance of the Jacobson radical was required. Until now the algebras simple with respect to an action of $H_4(-1)$ were the only example where the analog of Amitsur's conjecture was proved for an H-simple non-semisimple algebra [14]. In this article we prove the analog of Amitsur's conjecture for all finite dimensional $H_{m^2}(\zeta)$-simple algebras not necessarily semisimple (Section 4) assuming that the base field is algebraically closed and of characteristic 0.

1. Introduction

An algebra A over a field F is an *H-module algebra* for some Hopf algebra H if A is endowed with a homomorphism $H \to \mathrm{End}_F(A)$ such that $h(ab) = (h_{(1)}a)(h_{(2)}b)$ for all $h \in H$, $a, b \in A$. Here we use Sweedler's notation $\Delta h = h_{(1)} \otimes h_{(2)}$ where Δ is the comultiplication in H. We refer the reader to [8,16,17] for an account of Hopf algebras and algebras with Hopf algebra actions.

Let A be an H-module algebra for some Hopf algebra H over a field F. We say that A is *H-simple* if $A^2 \neq 0$ and A has no non-trivial two-sided H-invariant ideals.

Let $m \geqslant 2$ be an integer and let ζ be a primitive mth root of unity in a field F. (Such root exists in F only if $\mathrm{char}\, F \nmid m$.) Consider the algebra $H_{m^2}(\zeta)$ with unity generated by elements c and v satisfying the relations $c^m = 1$, $v^m = 0$, $vc = \zeta cv$. Note that $(c^i v^k)_{0 \leqslant i,k \leqslant m-1}$ is a basis of $H_{m^2}(\zeta)$. We introduce on $H_{m^2}(\zeta)$ a structure of a coalgebra by $\Delta(c) = c \otimes c$, $\Delta(v) = c \otimes v + v \otimes 1$, $\varepsilon(c) = 1$, $\varepsilon(v) = 0$. Then $H_{m^2}(\zeta)$ is a Hopf algebra with the antipode S where $S(c) = c^{-1}$ and $S(v) = -c^{-1}v$. The algebra $H_{m^2}(\zeta)$ is called a *Taft algebra*.

Remark. Note that if A is an $H_{m^2}(\zeta)$-module algebra, then the group $\langle c \rangle \cong \mathbb{Z}_m$ is acting on A by automorphisms. Every algebra A with a $\mathbb{Z}_m$-action by automorphisms is a $\mathbb{Z}_m$-graded algebra:

$$A^{(i)} = \left\{ a \in A \mid ca = \zeta^i a \right\},$$

$A^{(i)}A^{(k)} \subseteq A^{(i+k)}$. Conversely, if $A = \bigoplus_{i=0}^{m-1} A^{(i)}$ is a $\mathbb{Z}_m$-graded algebra, then $\mathbb{Z}_m$ is acting on A by automorphisms: $ca^{(i)} = \zeta^i a^{(i)}$ for all $a^{(i)} \in A^{(i)}$. Moreover, the notions of $\mathbb{Z}_m$-simple and $\mathbb{Z}_m$-graded-simple algebras are equivalent.

Remark. Theorems 5 and 6 of [5] imply that every $\mathbb{Z}_m$-grading on $M_n(F)$, where F is an algebraically closed field, is, up to a conjugation, *elementary*, i.e. there exist $g_1, g_2, \ldots, g_n \in \mathbb{Z}_m$ such that each matrix unit e_{ij} belongs to $A^{(g_i^{-1} g_j)}$. Rearranging rows and columns, we may assume that every $\mathbb{Z}_m$-action on $M_n(F)$ is defined by $ca = Q^{-1}aQ$ for some matrix

$$Q = \mathrm{diag}\{\underbrace{1, \ldots, 1}_{k_0}, \underbrace{\zeta, \ldots, \zeta}_{k_1}, \ldots, \underbrace{\zeta^{m-1}, \ldots, \zeta^{m-1}}_{k_{m-1}}\}.$$

2. Semisimple $H_{m^2}(\zeta)$-simple algebras

In this section we treat the case when an $H_{m^2}(\zeta)$-simple algebra A is semisimple.

Theorem 1. *Let A be a semisimple $H_{m^2}(\zeta)$-simple algebra over an algebraically closed field F. Then*

$$A \cong \underbrace{M_k(F) \oplus M_k(F) \oplus \cdots \oplus M_k(F)}_{t} \quad \textit{(direct sum of ideals)}$$