Journal of Pure and Applied Algebra 219 (2015) 3355–3371

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Rational points and orbits on the variety of elementary subalgebras

Jared Warner *

USC Dornsife, Department of Mathematics, 3620 S. Vermont Ave., KAP 104, Los Angeles, CA 90089-2532, USA

ARTICLE INFO

Article history:
Received 3 June 2014
Received in revised form 9 October 2014
Available online 4 November 2014
Communicated by D. Nakano

MSC:
17B45; 20G40

ABSTRACT

For G a connected, reductive group over an algebraically closed field k of large characteristic, we use the canonical Springer isomorphism between the nilpotent variety of $\mathfrak{g} := \mathrm{Lie}(G)$ and the unipotent variety of G to study the projective variety of elementary subalgebras of $\mathfrak{g}$ of rank r, denoted $\mathbb{E}(r, \mathfrak{g})$. In the case that G is defined over $\mathbb{F}_p$, we define the category of $\mathbb{F}_q$-expressible subalgebras of $\mathfrak{g}$ for $q = p^d$, and prove that this category is isomorphic to a subcategory of Quillen's category of elementary abelian subgroups of the finite Chevalley group $G(\mathbb{F}_q)$. This isomorphism of categories leads to a correspondence between G-orbits of $\mathbb{E}(r, \mathfrak{g})$ defined over $\mathbb{F}_q$ and G-conjugacy classes of certain elementary abelian subgroups of rank rd in $G(\mathbb{F}_q)$ which satisfy a closure property characterized by the Springer isomorphism. We use Magma to compute examples for $G = \mathrm{GL}_n$, $n \leq 5$.

In [2], J. Carlson, E. Friedlander, and J. Pevtsova initiated the study of $\mathbb{E}(r, \mathfrak{g})$, the projective variety of rank r elementary subalgebras of a restricted Lie algebra $\mathfrak{g}$. The authors demonstrate that the study of $\mathbb{E}(r, \mathfrak{g})$ informs the representation theory and cohomology of $\mathfrak{g}$. This is all reminiscent of the case of a finite group G, where the elementary abelian p-subgroups play a significant role in the story of the representation theory and cohomology of G, as first explored by Quillen in [11] and [12].

In this paper, we further explore the structure of $\mathbb{E}(r, \mathfrak{g})$ and its relationship with elementary abelian subgroups. Theorem 3.3 shows in the case that $\mathfrak{g}$ is the Lie algebra of a connected, reductive group G defined over $\mathbb{F}_p$, the category of $\mathbb{F}_q$-expressible subalgebras (Definitions 2.2 and 3.2) is isomorphic to a subcategory of Quillen's category of elementary abelian p-subgroups of $G(\mathbb{F}_q)$, where $q = p^d$. Specifically, we introduce the notion of an $\mathbb{F}_q$-linear subgroup (Definition 3.5), and we show in Corollary 3.9 that the $\mathbb{F}_q$-expressible subalgebras of rank r are in bijection with the $\mathbb{F}_q$-linear elementary abelian subgroups of rank rd in $G(\mathbb{F}_q)$. This bijection leads to Corollary 3.11, which allows us to compute the largest integer $R = R(\mathfrak{g})$ such that $\mathbb{E}(R, \mathfrak{g})$ is non-empty for a simple Lie algebra $\mathfrak{g}$. These values are presented in Table 1.

* Tel.: +1 805 698 4463.
E-mail address: hjwarner@usc.edu.

http://dx.doi.org/10.1016/j.jpaa.2014.10.019
0022-4049/

The results and definitions in Section 3 rely on the canonical Springer isomorphism $\sigma : \mathcal{N}(\mathfrak{g}) \to \mathcal{U}(G)$, which has been shown to exist under the hypotheses we assume in this paper, as detailed in [13,3,16], and [7]. Together with Lang's theorem, Theorem 3.3 implies Theorem 4.3, which establishes a natural bijection between the G-orbits of $\mathbb{E}(r, \mathfrak{g})$ defined over $\mathbb{F}_q$ and the G-conjugacy classes of $\mathbb{F}_q$-linear elementary abelian subgroups of rank rd in $G(\mathbb{F}_q)$. Example 4.8, due to R. Guralnick, shows that $\mathbb{E}(r, \mathfrak{g})$ may be an infinite union of G-orbits (in fact this is usually the case). However, Proposition 4.4 demonstrates that $\mathbb{E}(R(\mathfrak{g}), \mathfrak{g})$ is a finite union of orbits for all connected, reductive G such that (G, G) is an almost-direct product of simple groups of classical type. We believe that $\mathbb{E}(R(\mathfrak{g}), \mathfrak{g})$ is a finite union of orbits for all connected, reductive groups, and Proposition 4.4 reduces the verification of this belief to proving a claim about conjugacy classes of elementary abelian p-subgroups in $G(\mathbb{F}_q)$ for varying d and for exceptional simple groups G. Our interest in describing the G-orbits is motivated by the results of §6 in [2], where the authors construct algebraic vector bundles on G-orbits of $\mathbb{E}(r, \mathfrak{g})$ associated to a rational G-module M via the restriction of image, cokernel, and kernel sheaves.

Through personal communication with the author, E. Friedlander asked for conditions implying that $\mathbb{E}(r, \mathfrak{g})$ is irreducible. In the case that $\mathfrak{g} = \mathfrak{gl}_n$, Theorem 5.1 presents certain ordered pairs (r, n) for which $\mathbb{E}(r, \mathfrak{g})$ is irreducible. This theorem relies on previous results concerning the irreducibility of $\mathcal{C}_r(\mathcal{N}(\mathfrak{gl}_n))$, the variety of r-tuples of pair-wise commuting, nilpotent $n \times n$ matrices (see [8] for a nice summary of these results).

Finally, in Section 6, we compute a few examples for $G = \mathrm{GL}_n$. Some of the computations depend on Conjecture 6.1, which supposes the dimension of an orbit is related to the size of the corresponding G-conjugacy class. Eq. (6.2.1) computes the dimension of $\mathbb{E}(r, \mathfrak{gl}_n)$ for all (r, n) such that $\mathcal{C}_r(\mathcal{N}(\mathfrak{gl}_n))$ is irreducible, and surprisingly this equation agrees with computations of $\dim(\mathbb{E}(r, \mathfrak{gl}_n))$ even for ordered pairs where $\mathcal{C}_r(\mathcal{N}(\mathfrak{gl}_n))$ is known to be reducible. Proposition 6.3 computes the dimension of the open orbit defined by a regular nilpotent element, as first considered in Proposition 3.19 of [2]. For $n \leq 5$, we bound the number of G-orbits in $\mathbb{E}(r, \mathfrak{gl}_n)$ defined over $\mathbb{F}_q$ and compute their dimensions.

1. Review and preliminaries

Let k be an algebraically closed field of characteristic $p > 0$, and let G be a connected, reductive algebraic group over k, with Coxeter number $h = h(G)$. Following §2 in [19], we let $\pi = \pi(G)$ denote the fundamental group of $G' = (G, G)$. We will often require that p satisfies the following two conditions, which will be collectively referred to as condition $(\star)$:

$$(1) \quad p \geq h, \qquad (2) \quad p \nmid |\pi| \tag{$\star$}$$

We make three remarks about condition $(\star)$. First, (1) implies (2) in all cases except when $p = h$ and G' has an adjoint component of type A. Second, (2) is equivalent to the separability of the universal cover $G'_{sc} \to G'$ [19, §2.4]. For example, the canonical map $\mathrm{SL}_p \to \mathrm{PSL}_p$ is not separable in characteristic p, so we must exclude the case $G = \mathrm{PSL}_p$. Third, $(\star)$ implies that p is non-torsion for G (cf. §2 in [9]), which we require to use Theorem 2.2 of [9] in our proof of Theorem 1.3.

The unipotent elements of G form an irreducible closed subvariety of G, denoted $\mathcal{U}(G)$, and G acts by conjugation on $\mathcal{U}(G)$. In the Lie algebra setting, the nilpotent elements of $\mathfrak{g} := \mathrm{Lie}(G)$ also form an irreducible closed subvariety of $\mathfrak{g}$, denoted $\mathcal{N}(\mathfrak{g})$, and $\mathcal{N}(\mathfrak{g})$ is a G-variety under the adjoint action of G on $\mathfrak{g}$. The main tool we will use to translate information between the group and Lie algebra settings will be a well-behaved Springer isomorphism.

Definition 1.1. A *Springer isomorphism* is a G-equivariant isomorphism of algebraic varieties $\sigma : \mathcal{N}(\mathfrak{g}) \to \mathcal{U}(G)$.