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On linear shift representations

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ABSTRACT

We introduce and develop the concept of (linear) *shift representation*. This derives from a certain action on 2-cocycle groups that preserves both cohomological equivalence and orthogonality for cocyclic designs, discovered by K.J. Horadam. Detailed information about fixed point spaces and reducibility is given. We also discuss results of computational experiments, including the calculation of shift orbit structure and searching for orthogonal cocycles.

1. Introduction

In [10], Horadam defines the *shift action* of a finite group G on the set of its 2-cocycles with trivial coefficients in an abelian group U. This action originates from equivalence of well-known objects in cocyclic design theory (see [2, Chapters 13, 15] and [9, Chapter 7]).

For example, let $U = \langle -1 \rangle \cong \mathrm{C}_2$; a cocycle $\psi : G \times G \to U$ is *orthogonal* if $H = [\psi(g,h)]_{g,h \in G}$ is a Hadamard matrix, i.e., $HH^\top = nI_n$ where $n = |G|$. Any such cocycle yields a relative difference set in the corresponding central extension of U by G with forbidden subgroup U, and vice versa [3]. These extensions, called *Hadamard groups*, were studied by Ito [11] using sophisticated algebraic techniques (see also [7]).

Orthogonal cocycles have diverse applications [9]. Moreover, de Launey and Horadam conjecture that there exists a cocyclic Hadamard matrix at every order $n = 4t$ [9, p. 134]. It is unfortunate, then, that orthogonality and cohomological equivalence are incompatible: cocycles from the same cohomology class as an orthogonal cocycle need not themselves be orthogonal. A naive search for orthogonal cocycles would therefore run over the full cocycle space, whose size depends exponentially on $|G|$, rather than over the very much smaller space of cohomology classes. On the other hand, shift action respects both orthogonality and cohomological equivalence. That is, cocycles lying in the same shift orbit are cohomologous, and they are all orthogonal if any one of them is.

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Shift action is consequently an important tool in the study of cocyclic pairwise combinatorial designs and their applications. We introduce and develop a representation-theoretic variant of this action, which allows us to treat it in a practical way: via matrix groups acting on an underlying vector space of reasonable dimension. To this end, we provide a comprehensive description of complete reducibility and fixed points for shift representations. Our paper extends the reach of previous work such as [12], and is a starting point for further investigation of shift action in a computationally tractable setting.

We now summarize the content of the paper. In Section 2 we prove elementary facts about shift representations. The main results of Section 3 are a determination of fixed points under shift action in the full cocycle space, and a bound on the dimension of the fixed coboundary space. We thereby solve most of Research Problem 55 (1) in [9]. Some relevant linear group theory is then given in Section 4. This serves as background for Section 5, where we establish that a shift representation is hardly ever completely reducible. In fact, we provide criteria for deciding irreducibility and complete reducibility. As an illustration of the practical nature of our approach, in the final section we describe new results obtained from our Magma [1] implementation of procedures to compute with shift representations. Open questions arising from the computational work are posed.

We remark that the machinery set up in this paper has been applied successfully in a recent classification of Butson Hadamard matrices of order n over pth roots of unity, for p prime and $np \leq 100$ [5].

2. Preliminaries

Let G and U be finite non-trivial groups, with U abelian. A map $\phi : G^n \to U$ is *normalized* if $\phi(x) = 1$ whenever x has 1 in at least one component. We denote by $F(G^n, U)$ the abelian group of normalized maps $G^n \to U$ under pointwise multiplication. A *cocycle* is an element ψ of $F(G^2, U)$ such that

$$\psi(g,h)\psi(gh,k) = \psi(g,hk)\psi(h,k) \quad \forall g,h,k \in G. \tag{1}$$

The cocycles form a subgroup $Z(G,U) \leq F(G^2, U)$. If $\phi \in F(G,U)$ then $\partial\phi \in Z(G,U)$ defined by

$$\partial\phi(g,h) = \phi(g)^{-1}\phi(h)^{-1}\phi(gh)$$

is a *coboundary*. The map $\partial : F(G,U) \to Z(G,U)$ is a homomorphism with kernel $\mathrm{Hom}(G,U) \cong \mathrm{Hom}(G/G', U)$ where $G' = [G,G]$. Put $\mathrm{im}\,\partial = B(G,U)$. The elements of $H(G,U) = Z(G,U)/B(G,U)$ are *cohomology (equivalence) classes*; equivalent elements of $Z(G,U)$ are *cohomologous*. If $|G|$, $|U|$ are coprime (for example) then $H(G,U) = 0$.

For $\psi \in Z(G,U)$ and $a,g \in G$, set $\psi_a(g) = \psi(a,g)$. Using (1) we verify that $\psi_a\psi_b(\partial(\psi_a))_b = \psi_{ab}$, so

$$\psi a = \psi\partial(\psi_a) \tag{2}$$

defines an action of G on $Z(G,U)$ [10, Section 3]. This *shift action* obviously preserves cohomological equivalence.

Let Γ be the permutation representation $G \to \mathrm{Sym}(Z(G,U))$ associated to (2). If $S \subseteq Z(G,U)$ is $\Gamma(G)$-invariant then Γ_S will denote the restricted representation of G in $\mathrm{Sym}(S)$.

Lemma 2.1. *Suppose that S is a $\Gamma(G)$-invariant subgroup of $Z(G,U)$. Then Γ_S is a homomorphism $G \to \mathrm{Aut}(S)$.*