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Plane curves containing a star configuration

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ABSTRACT

Given a collection of l general lines $\ell_1, \dots, \ell_l$ in $\mathbb{P}^2$, the star configuration $\mathbb{X}(l)$ is the set of points constructed from all pairwise intersections of these lines. For each non-negative integer d, we compute the dimension of the family of curves of degree d that contain a star configuration.

1. Introduction

Throughout this paper, $S = k[x_0, x_1, x_2]$ with k an algebraically closed field. Given any linear form $L \in S$, we let ℓ denote the corresponding line in $\mathbb{P}^2$. Let $\ell_1, \dots, \ell_l$ be a set of l distinct lines in $\mathbb{P}^2$. Classically, the union of these l lines is called an *l-lateral*. A *complete l-lateral* is the union of l lines, such that $\ell_i \cap \ell_j \cap \ell_k = \emptyset$ for all triples $\{i, j, k\} \subseteq \{1, \dots, l\}$. We say that a plane curve has an inscribed l-lateral if it contains the $\binom{l}{2}$ vertices of the l-lateral, that is, the $\binom{l}{2}$ points formed by taking all possible intersections of lines.

Let $\mathbb{X}(l)$ denote the collection of points formed by taking all possible intersections of a complete l-lateral. Such a collection of points is sometimes called a *star configuration*. The name star configuration arises from the fact that a complete 5-lateral that contains an $\mathbb{X}(5)$ resembles a star. These special configurations, and their generalizations in $\mathbb{P}^n$, have recently risen in prominence due, in part, to the fact that they have nice algebraic properties (e.g., the minimal generators are products of linear forms), but at the same time exhibit some extremal properties (e.g., the work of Bocci and Harbourne [4] which compares symbolic and regular powers of ideals). The papers [1,3,10–12,17,18] are some of the papers that have contributed to our understanding of star configurations. Because this paper is related to our previous papers (see [5,6]) on star

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configurations, we shall prefer to use the terminology of star configurations as opposed to the language of l-laterals found in [8,15,16]. Moreover, star configurations may better lend themselves to higher dimensional generalizations (see our concluding remarks). Moving forward, we will primarily refer to the family of curves in $\mathbb{P}^2$ of degree d that "contain a star configuration $\mathbb{X}(l)$" as opposed to "contain an inscribed l-lateral".

In this paper we compute the dimension of the family of curves in $\mathbb{P}^2$ of degree d that contain a star configuration $\mathbb{X}(l)$, or equivalently, an inscribed l-lateral. More precisely, if $l > 2$, consider the quasi-projective variety

$$\mathcal{D}_l \subseteq \underbrace{\check{\mathbb{P}}^2 \times \cdots \times \check{\mathbb{P}}^2}_{l}$$

where $(\ell_1, \ldots, \ell_l) \in \mathcal{D}_l$ if and only if no three of the lines meet at a point; here $\check{\mathbb{P}}^2$ denotes the dual projective space. Notice that $\mathcal{D}_l$ can be seen as a parameter space for star configuration set of points obtained by intersecting l general lines. With a slight abuse of notation, we will often write $\mathbb{X}(l) \in \mathcal{D}_l$, thus identifying a star configuration with the unique set of lines defining it.

We construct the following incidence correspondence

$$\Sigma_{d,l} = \left\{ \big(\mathcal{C}, \mathbb{X}(l)\big) : \mathcal{C} \supseteq \mathbb{X}(l) \right\} \subseteq \mathbb{P}S_d \times \mathcal{D}_l. \tag{$\star$}$$

Letting $\phi_{d,l} : \Sigma_{d,l} \to \mathbb{P}S_d$ denote the natural projection map, we define the *locus of degree d curves containing a star configuration* $\mathbb{X}(l)$, denoted $\mathcal{S}(d, l)$, to be $\mathcal{S}(d, l) = \phi_{d,l}(\Sigma_{d,l})$. We then prove the following result about the dimension of the locus.

Theorem 1.1. *Let $d \geq 0$ and $l \geq 2$ be integers. Then $\mathcal{S}(d, l) = \emptyset$ if $d < l - 1$, and*

$$\dim \mathcal{S}(d, l) = \begin{cases} \binom{d+2}{2} - 1 & \text{if } d \geq l - 1 \text{ and } l = 2, 3, 4 \\ \binom{d+2}{2} - 2 & \text{if } d = 4 \text{ and } l = 5 \\ \binom{d+2}{2} - 1 & \text{if } d \geq 5 \text{ and } l = 5 \\ \binom{d+2}{2} - \binom{l}{2} + 2l - 1 & \text{if } d \geq l - 1 \text{ and } l \geq 6. \end{cases}$$

Theorem 1.1 complements our previous work [5,6] which showed that the generic degree d plane curve contains a star configuration $\mathbb{X}(l)$ if and only if the projection map $\phi_{d,l}$ is dominant which happens if and only if $\dim \mathcal{S}(d, l) = \binom{d+2}{2} - 1$. The tuples (d, l) for which $\dim \mathcal{S}(d, l) = \binom{d+2}{2} - 1$ are therefore precisely the tuples described in [6, Theorem 6.3]. The fact that $\mathcal{S}(d, l) = \emptyset$ for $d < l - 1$ comes from Bezout's Theorem (see Remark 2.2).

It therefore suffices to focus on proving Theorem 1.1 for the pairs $(4, 5)$ and (d, l) with $d \geq l - 1$ and $l \geq 6$. Our strategy for the pairs $(d, l) \neq (4, 5)$ is to first translate the problem into computing the dimension of a graded ideal constructed from the linear forms $L_1, \ldots, L_l$ in a particular degree. This enables us to reduce the problem to computing the rank of a particular matrix. We use the notion of Lüroth quartics to deal with the pair $(d, l) = (4, 5)$. Note that $\mathcal{S}(4, 5)$ is the only $\mathcal{S}(d, l)$ whose dimension differs from the expected one.

The family $\mathcal{S}(d, d + 1)$ was also studied by Barth [2, Application 2], who also computed their dimensions, and by Ellingsrud, Le Potier, and Strømme [9, Section 4], who raised the still-open question of computing the cardinality of the fibres of $\phi_{d,d+1}$.

Our paper is structured as follows. In Section 2 we recall the relevant facts about star configurations. We also translate our problem into a new algebraic question, and we compute $\dim \mathcal{S}(4, 5)$. In Section 3 we prove Theorem 1.1 for all tuples $(d, 6)$ with $d \geq 5$. The results of this section provide a base case for the arguments of Section 4. We conclude with remarks about the higher dimensional analog of this problem.