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Left-symmetric conformal algebras and vertex algebras ☆

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ABSTRACT

A vertex algebra is an algebraic counterpart of a two-dimensional conformal field theory. By an equivalent characterization of vertex algebra using Lie conformal algebra and left-symmetric algebra given by Bakalov and Kac in [3], in studying vertex algebra, we have to deal with such a question: Do there exist compatible left-symmetric algebra structures on a class of special Lie algebras named formal distribution Lie algebras? In this paper, we study this question. We introduce the definitions of left-symmetric conformal algebra and Novikov conformal algebra. Many examples of these algebras are obtained. As an application, we present a construction of vertex algebra using left-symmetric conformal algebra.

1. Introduction

Two-dimensional conformal field theory plays an important role in physics as providing models of quantum field theory. A vertex algebra is a rigorous mathematical definition of the chiral part of a 2-dimensional quantum field theory studied intensively by physicists since the landmark paper [8]. Axiomatically, the notion of a vertex algebra was introduced by R. Borcherds in [6]. It develops in conjunction with string theory in theoretical physics and with the theory of "monstrous moonshine" and infinite-dimensional Lie algebra theory in mathematics (see [6,23]).

Throughout this paper, denote by $\mathbb{C}$ the field of complex numbers; $\mathbf{N}$, the set of natural numbers, i.e. $\mathbf{N} = \{0, 1, 2, \cdots\}$; $\mathbb{Z}$, the set of all integers. And, tensors over $\mathbb{C}$ are denoted by $\otimes$.

Let V be a vector space over $\mathbb{C}$. In the theory of vertex algebras [6,28], a *field* on V is a series of the form $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ where $a_{(n)} \in \operatorname{End} V$, and for each $v \in V$ one has $a_{(n)}(v) = 0$ for sufficiently large n.

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Definition 1.1. (See [28].) A *vertex algebra* is a complex vector space V endowed with the *state-field correspondence* Y which is a $\mathbb{C}$-linear map of V to the space of fields

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \operatorname{End} V,$$

a *vacuum element* $|0\rangle$ and an operator $T \in \operatorname{End} V$ satisfying the following axioms ($a, b \in V$):

(translation covariance): T satisfies

$$\big[T, Y(a,z)\big] = Y(Ta,z) = \frac{d}{dz} Y(a,z),$$

(vacuum axiom): $Y(|0\rangle, z) = id_V, Y(a,z)|0\rangle|_{z=0} = a$,
(locality): $(z-w)^{N_{a,b}}[Y(a,z), Y(b,w)] = 0$ for some sufficiently large non-negative integer $N_{a,b}$.

The locality condition for two fields is equivalent to the commutative formula (see [28])

$$\big[a(z), b(w)\big] = \sum_{j=0}^{N_{a,b}-1} c_j(w) \frac{\partial_w^j \delta(z-w)}{j!},$$

for some new field $c_j(w)$ and $\delta(z-w) = z^{-1}\sum_{n\in\mathbb{Z}}(\frac{w}{z})^n$. The operator product expansion (OPE) can be written symbolically

$$a(z)b(w) = \sum_{j\in\mathbb{Z}} c_j(w)(z-w)^{-j-1}.$$

The new field c_j is called the j-th *product* of a, b and denoted by $a_{(j)}b$. $a_{(-1)}b$ is called the *Wick product (=normally ordered product)*. The j-th products satisfy the *Borcherds identity* (see [6] or [28]).

The commutator of two fields is encoded by the singular part of their OPE and is uniquely determined by their j-th products for $j \geq 0$. Define $\operatorname{Res}_z a(z) = a_{(0)}$. The λ-bracket

$$[a_\lambda b] = \operatorname{Res}_z e^{z\lambda} a(z) b = \sum_{j=0}^{N_{a,b}-1} \lambda^j \frac{a_{(j)}b}{j!}$$

satisfies the axioms of a *Lie conformal algebra* introduced by Kac in [28,29]. It is also known as vertex Lie algebra in [19,39] or Lie pseudoalgebra [2] over $\mathbb{C}[T]$ (the definition can be found in Section 2). In addition, Lie conformal algebras have close connections to Hamiltonian formalism in the theory of nonlinear evolution equations (see the book [20] and references therein, and also [24,48,47] and many other papers). The structure theory [18], representation theory [13,14] and cohomology theory [4] of finite Lie conformal algebras have been developed.

Lie conformal algebra is a useful tool to study vertex algebra. A vertex algebra is naturally a Lie conformal algebra. In addition, a Lie conformal algebra can bring a universal enveloping vertex algebra. This functor is adjoint to the forgetful functor from vertex algebras to Lie conformal algebras. Many interesting vertex algebras are simple quotients of the universal enveloping algebras associated with some Lie conformal algebras.

On the other hand, we should introduce a class of non-associative algebras named left-symmetric algebra. A *left-symmetric algebra* A is a vector space over $\mathbb{C}$ with the operation "$\circ$" satisfying

$$(a \circ b) \circ c - a \circ (b \circ c) = (b \circ a) \circ c - b \circ (a \circ c), \quad a, b, c \in A. \tag{1.1}$$