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Resolution algorithms and deformations

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ABSTRACT

An algorithm for resolution of singularities in characteristic zero is described. It is expressed in terms of multi-ideals, that essentially are defined as a finite sequence of pairs, each one consisting of a sheaf of ideals and a positive integer. This approach is particularly simple and seems suitable for applications to a good theory of simultaneous algorithmic resolution of singularities, specially for families parametrized by the spectrum of an artinian ring.

0. Introduction

Algorithmic, constructive, or canonical methods to resolve singularities of algebraic varieties attempt to clarify and simplify the original proof of the main desingularization theorem [10]. These are programs to eliminate the singularities of an algebraic variety by means of a sequence of blowing-ups with well determined regular centers. So far these algorithms proceed indirectly working primarily with some auxiliary objects, such as *marked ideals*, *basic objects* or *presentations*. Suitable resolution algorithms for such objects imply similar results for varieties. At present there are several algorithmic processes to desingularize algebraic varieties over fields of characteristic zero ([1–5,7,12,14,16,17], etc.).

Algorithmic resolutions (in characteristic zero) being available nowadays, it becomes reasonable to investigate the possibility to simultaneously resolve the members of a family of varieties, or their related objects, using a given resolution algorithm. Results in this direction were obtained in [6], in the case where the parameter scheme is regular. The general case was studied in [13] and [15], using essentially the algorithm of [7].

More precisely, in [13] we consider the crucial case where the parameter space is the spectrum of an artinian ring, i.e., that of an infinitesimal deformation of an object over a field. Most of the discussion of that paper is in the context of *basic objects*, i.e., systems $(W,(I,b),E)$ where W is a variety smooth over a characteristic zero field k, I is a coherent sheaf of $\mathcal{O}_W$-ideals, b a positive integer, and E a set of regular divisors of W with normal crossings. To develop a reasonable theory of *simultaneous resolution* or

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equiresolution, we try to imitate what the algorithm does when the base is a field. First, we introduce basic objects over an artinian ring and a notion of permissible centers in this context, i.e., the centers we allow in our blow-ups. Given a basic object B over an artinian ring, we have a naturally defined closed fiber $B^{(0)}$, which is a basic object over a field. Then we attempt to "naturally extend" the permissible centers used in the algorithmic resolution of $B^{(0)}$ to permissible centers of B and its transforms. If this can be done for all the centers used in the algorithmic resolution of $B^{(0)}$, we say that B is *algorithmically equisolvable*.

In [15] we use these results as the basis of an equiresolution theory for families parameterized by more general schemes, such as arbitrary algebraic schemes over characteristic zero fields. We prove that if the parameter space is regular the new notion of equiresolution agrees with those discussed in [6], which are geometrically more intuitive but not directly applicable in the presence of nilpotents.

However, there is an aspect of the theory of [13] which, perhaps, is not entirely satisfactory. To explain it, let us briefly recall an important feature of the algorithm of [7], shared by all those already mentioned.

Working over a field, the algorithmic resolution process requires an "inductive step". For instance, given a basic object $B = (W, (I, b), E)$ satisfying certain conditions, to obtain the first center to blow-up in the resolution process, we proceed as follows. Substitute B (near a point $x \in W$) by another "inductive" basic object $B^\star = (Z, (J, c), E^\star)$, where Z is a suitable hypersurface, defined on an appropriate neighborhood U of x. Since $\dim Z < \dim W = d$, if we assume, by induction, that algorithmic resolution is available when the dimension is $< d$, we have a first (or zeroth) algorithmic resolution center for $B^\star$. This is a closed subscheme of U, i.e., a locally closed subscheme of W. Since these centers are defined just locally, there is a glueing problem. But it can be proved that they agree on intersections to produce a closed subscheme C of W, which is a B-permissible center. This is the first algorithmic center for B.

In [13], working over an artinian ring A instead of over a field, we have shown that it is possible, to imitate, to some extent the constructions of [7]. Indeed, if B is a basic object over A with special fiber $B^{(0)}$, we can impose reasonable conditions so that when $B^{(0)}$ is in the inductive situation, an analog for $B^\star$ is defined. If, by induction on the dimension, $B^\star$ is algorithmically equisolvable, then we have a first algorithmic equiresolution center C for $B^\star$. This is a closed subscheme of W, which should be the first center for B. But there is a drawback.

Unfortunately, sometimes this subscheme C of W might not be a permissible center for B, because an equality of orders of certain ideals required for such centers may fail. Example 6.15 in [13] shows that this is possible. Thus, a condition to be imposed in the definition of equiresolution is that this center C be permissible for B, so that this notion is not strictly recursive.

This "pathology" is due to the fact that working over artinian rings, the local rings of our schemes have nilpotents. The algorithm used in [13] involves certain constructions like the *coefficient ideal* $C(I)$ and the auxiliary object B''_s described in [7], which require to take powers of ideals. Since our rings are not reduced, certain powers of elements that, by analogy with the classical case, should not be zero, sometimes vanish causing the mentioned difficulties.

Thus, for application to deformations, it seems convenient to use a resolution algorithm which avoids powers of ideals, at least in the constructions related to the crucial inductive step. This paper proposes such a resolution algorithm. The new algorithm is an adaptation of that of [7] (as presented in [14]) but, instead of basic objects, it involves *multi-ideals* that essentially are systems $(W, (I_1, b_1), \dots, (I_n, b_n), E)$ where, for each index i, $(W, (I_i, b_i), E)$ is a basic object.

In the "classical" context, where we work over a base field, the algorithm we present does not yield anything essentially new, except that perhaps certain points are a little easier to verify. But the use of multi-ideals allows us to avoid certain constructions involving taking powers of elements in our local rings, and this becomes relevant working over more general rings. Indeed, in the last section we explain how to partially generalize the new algorithm to the situation where the base is a suitable artinian ring rather than a field. In particular, we show that if B is a suitable basic object we have an inductive multi-ideal $B^\star$ (of smaller