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The Kaplansky radical of a quadratic field extension

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ABSTRACT

The radical of a field consists of all nonzero elements that are represented by every binary quadratic form representing 1. Here, the radical is studied in relation to local–global principles, and further in its behavior under quadratic field extensions. In particular, an example of a quadratic field extension is constructed where the natural analogue to the square-class exact sequence for the radical fails to be exact. This disproves a conjecture of Kijima and Nishi.

1. Introduction

Let K be a field of characteristic different from 2. Let $K^\times$ denote the multiplicative group of K, $\sum K^{\times 2}$ the subgroup of nonzero sums of squares in K, and $D_K\langle 1, a\rangle$ the subgroup of $K^\times$ consisting of the nonzero elements represented by the binary quadratic form $X^2 + aY^2$, for any $a \in K^\times$. The object of study in this article is the subgroup

$$\mathrm{R}(K) = \bigcap_{a\in K^\times} D_K\langle 1, a\rangle$$

of $K^\times$, called the (*Kaplansky*) *radical* of K. This object was first studied by I. Kaplansky for fields over which there exists a unique quaternion division algebra [7]. It was investigated in more generality by C.M. Cordes [4], who baptized it the *Kaplansky radical* and observed that in several statements about quadratic forms over K one can replace $K^{\times 2}$ by $\mathrm{R}(K)$. We refer to [11, Chapter XII, Sections 6 & 7] for an introduction to the Kaplansky radical. By [11, Chapter XII, (6.1)] the radical is further characterized as $\mathrm{R}(K) = \{c \in K^\times \mid D_K\langle 1, -c\rangle = K^\times\}$.

In this article we continue the study of the radical. In Section 2 we consider the position of the radical within the inclusions $K^{\times 2} \subseteq \mathrm{R}(K) \subseteq \sum K^{\times 2}$. In Section 3 we study fields satisfying a local–global principle for quadratic forms and derive a determination of the radical as the set of elements that are locally squares. In Section 4 we revisit the behavior of the radical under quadratic field extensions and disprove a conjecture by D. Kijima and M. Nishi discussed in [8], [9], and [6].

2. Position of the radical

We have the inclusions $K^{\times 2} \subseteq \mathrm{R}(K) \subseteq D_K\langle 1, 1\rangle \subseteq \sum K^{\times 2}$. We first consider the two extremal cases for the position of the radical with respect to these inclusions. We say that K is *radical-free* if $\mathrm{R}(K) = K^{\times 2}$.

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Proposition 2.1. *Assume that* $|K^\times/K^{\times 2}| \geqslant 4$ *and there exists* $t \in K^\times$ *such that* $D_K\langle 1,t\rangle = K^{\times 2} \cup tK^{\times 2}$ *and* $D_K\langle 1,-t\rangle = K^{\times 2} \cup -tK^{\times 2}$. *Then K is radical-free.*

Proof. We may choose an element $a \in K^\times \setminus (K^{\times 2} \cup tK^{\times 2})$. Then $a \notin D_K\langle 1,t\rangle$ and thus $-t \notin D_K\langle 1,-a\rangle$, whereby $\mathrm{R}(K) \subseteq D_K\langle 1,-t\rangle \cap D_K\langle 1,-a\rangle = K^{\times 2}$. $\square$

By a $\mathbb{Z}$-*valuation* we mean a valuation with value group $\mathbb{Z}$. For a $\mathbb{Z}$-valuation v on K we denote by K_v the corresponding completion.

Corollary 2.2. *Assume that K is henselian with respect to a* $\mathbb{Z}$*-valuation whose residue field is of characteristic different from* 2 *and not quadratically closed. Then K is radical-free.*

Proof. It follows from the hypotheses that $|K^\times/K^{\times 2}| \geqslant 4$. Moreover, any $t \in K^\times$ that has odd value with respect to the given valuation will be such that $D_K\langle 1,t\rangle = K^{\times 2} \cup tK^{\times 2}$ and $D_K\langle 1,-t\rangle = K^{\times 2} \cup -tK^{\times 2}$. Hence, the statement follows from (2.1). $\square$

By [11, Chapter XII, Section 6], if K is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ for a prime number p, then K is radical-free; for $p \neq 2$ this can be seen from (2.2).

Proposition 2.3. *The following are equivalent*:

(i) $\mathrm{R}(K) = \sum K^{\times 2}$;
(ii) $\mathrm{R}(K) = D_K\langle 1,1\rangle$;
(iii) $I_t^2 K = 0$;
(iv) *every torsion* 2*-fold Pfister form over K is hyperbolic.*

Proof. This follows from [11, Chapter XI, (4.1) and (4.5)] for $n = 2$. $\square$

Condition (iv) corresponds to Property (A_2) in the terminology of [5], treated also in [11, Chapter XI, Section 4]. Following [9] we say that the field K is *quasi-pythagorean* if it satisfies the equivalent conditions in (2.3). By [11, Chapter XI, (6.26)] this is further equivalent to having that the u-invariant of K is at most 2. For example, by [11, Chapter XI, (4.10)], any extension of transcendence degree one of a real closed field is quasi-pythagorean.

In [4] Cordes gave an example of a field K with $K^{\times 2} \subsetneq \mathrm{R}(K) \subsetneq \sum K^{\times 2}$ and asked whether one can have such examples where $K^\times/K^{\times 2}$ is finite. M. Kula [10] and L. Berman [3] independently constructed such examples. We give another example where K is a nonreal algebraic extension of $\mathbb{Q}$ having 8 square classes.

Example 2.4. The integers $-2, -5$ and 7 are squares in $\mathbb{Q}_3$. Hence, $\mathbb{Q}_3$ contains the field $\mathbb{Q}(\sqrt{-2}, \sqrt{-5})$. Moreover, 7 is not a square in $\mathbb{Q}(\sqrt{-2}, \sqrt{-5})$. Consider the set of subfields of $\mathbb{Q}_3$ that are algebraic extensions of $\mathbb{Q}(\sqrt{-2}, \sqrt{-5})$ and in which 7 is not a square. By Zorn's Lemma, we may choose a maximal element K in this set. Then K is a field whose unique quadratic extension contained in $\mathbb{Q}_3$ is $K(\sqrt{7})$. As the four square classes of $\mathbb{Q}_3$ are represented by $1, 2, 3$ and 6, it follows that the classes of $2, 3, 7$ form an $\mathbb{F}_2$-basis of the square class group $K^\times/K^{\times 2}$. In particular $|K^\times/K^{\times 2}| = 8$.

As $\mathbb{Q}_3^\times = K^\times \mathbb{Q}_3^{\times 2}$ we conclude that $\mathrm{R}(K) \subseteq \mathrm{R}(\mathbb{Q}_3)$. As $\mathbb{Q}_3$ is radical-free, we obtain that $\mathrm{R}(K) \subseteq K^\times \cap \mathbb{Q}_3^{\times 2} = K^{\times 2} \cup 7K^{\times 2}$. Since $2 = 3^2 - 7$, $3 = (\sqrt{-2}\cdot\sqrt{-5})^2 - 7$ and $2\cdot 3\cdot 7 = 7^2 - 7$, we see that $D_K\langle 1,-7\rangle = K^\times$. This shows that $\mathrm{R}(K) = K^{\times 2} \cup 7K^{\times 2}$.

The number of square classes in (2.4) is minimal for having a nontrivial radical, by the following statement.

Proposition 2.5. *If* $K^{\times 2} \subsetneq \mathrm{R}(K) \subsetneq \sum K^{\times 2}$ *then* $|K^\times/K^{\times 2}| \geqslant 8$.

Proof. By [11, Chapter XII, (6.10)], if $\mathrm{R}(K)$ has index two in $K^\times$, then K is real and thus $\mathrm{R}(K) = \sum K^{\times 2}$. Hence, if $\mathrm{R}(K) \subsetneq \sum K^{\times 2}$ then $|K^\times/\mathrm{R}(K)| \geqslant 4$. $\square$

3. The radical as the group of local squares

In certain fields satisfying a local–global principle for isotropy of quadratic forms, the radical consists of the elements that are squares locally.