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Generalized weakly symmetric rings ☆

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ABSTRACT

A ring R is defined to be *GWS* if $abc = 0$ implies $bac \subseteq N(R)$ for $a, b, c \in R$, where $N(R)$ stands for the set of nilpotent elements of R. Since reduced rings and central symmetric rings are *GWS*, we study sufficient conditions for *GWS* rings to be reduced and central symmetric. We prove that a ring R is *GWS* if and only if the $n \times n$ upper triangular matrices ring $U_n(R, R)$ is *GWS* for any positive integer n. It is proven that *GWS* rings are directly finite and left min-abel. For a *GWS* ring R, R is a strongly regular ring if and only if R is a von Neumann regular ring if and only if R is a left *SF* ring and $J(R) = 0$; R is an exchange ring if and only if R is a clean ring. Finally, we show that *GWS* exchange rings have stable range 1 and a *GWS* semiperiodic ring R with $N(R) \neq J(R)$ is commutative.

1. Introduction

Throughout this article, all rings considered are associative with identity, and all modules are unital, the symbols $J(R)$, $N(R)$, $U(R)$, $E(R)$, $Z(R)$ and $Max_l(R)$ will stand respectively for the Jacobson radical, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements, the center and the set of all maximal left ideals of R. For any nonempty subset X of a ring R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilator of X and the left annihilator of X, respectively.

A ring R is called

(1) *reduced* if $N(R) = 0$;
(2) *symmetric* if $abc = 0$ implies $bac = 0$ for $a, b, c \in R$;
(3) *Abel* if $E(R) \subseteq Z(R)$;
(4) *left quasi-duo* if every maximal left ideal of R is an ideal;
(5) *MELT* if every essential maximal left ideal of R is an ideal.

Symmetric rings are defined by Lambek in [10]. In [8], this concept is extended to the *central symmetric* ring, that is, if $abc = 0$ implies $bac \in Z(R)$.

A ring R is called *generalized weakly symmetric* or *GWS* if $abc = 0$ implies $bac \in N(R)$. In this paper, we show that *GWS* rings are a proper generalization of central symmetric rings. Theorem 2.7 shows that a ring R is *GWS* if and only if the $n \times n$ upper triangular matrix ring over R is *GWS* for each $n \geqslant 1$.

Let R be a ring and $e \in E(R)$. e is called *left minimal idempotent* if Re is a minimal left ideal of R. We write $ME_l(R)$ for the set of all left minimal idempotents of R. A ring R is called *left min-abel* if $(1 - e)Re = 0$ for each $e \in ME_l(R)$.

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In [14, Theorem 1.2], it is shown that a ring R is left quasi-duo if and only if it is a left min-abel *MELT* ring. The study of left min-abel rings appears in [14], [16] and [17]. Theorem 2.13 shows that *GWS* rings are left min-abel.

Following [11], an element a of a ring R is called clean if a is a sum of a unit and an idempotent of R, and a is said to be exchange if there exists $e \in E(R)$ such that $e \in aR$ and $1-e \in (1-a)R$. A ring R is called clean if every element of R is clean, and R is said to be exchange if every element of R is exchange. According to [11], clean rings are always exchange, but the converse is not true, in general. In [19], it is shown that left quasi-duo exchange rings are clean; In [20], it is shown that Abel exchange rings are clean; In [16], it is shown that quasi-normal exchange rings are clean; In [17], it is shown that weakly normal exchange rings are clean. Theorem 4.1 shows that *GWS* exchange rings are clean.

Following [3], a ring R is said to be *semiperiodic* if for each $x \in R \backslash (J(R) \cup Z(R))$, there exist $m, n \in \mathbb{Z}$, of opposite parity, such that $x^n - x^m \in N(R)$. Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain non-nil periodic rings. In [3], it is shown that reduced semiperiodic rings are commutative. Theorem 5.3 shows that a *GWS* semiperiodic ring R with $N(R) \neq J(R)$ is commutative.

2. Generalized weakly symmetric rings

Definition 2.1. A ring R is called *generalized weakly symmetric* (*GWS*) if for any $a, b, c \in R$, $abc = 0$ implies $bac \in N(R)$.

All commutative rings, reduced rings and symmetric rings are *GWS*. One may suspect that *GWS* rings are symmetric. But the following example erases the possibility.

Example 2.2. Let D be a field and $R = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$. Then R is a *GWS* ring. But R is not symmetric.

Following [8], a ring R is called *central symmetric* if for any $a, b, c \in R$, $abc = 0$ implies $bac \in Z(R)$. Clearly, symmetric rings are central symmetric.

Proposition 2.3. *Central symmetric rings are GWS.*

Proof. Let R be a central symmetric ring and $abc = 0$. Then $(ab)(cx)1 = 0$ and $(ya)bc = 0$ for any $x, y \in R$. Hence $cxab \in Z(R)$ and $byac \in Z(R)$. Clearly, $(bac)^4 = bacbac(bac)bac = bacba(bac)cbac = (b(acbab)ac)cbac = cba(b(acbab)ac)c = cbaba(cbab)acc = cbab(cbab)aacc = 0$, so $bac \in N(R)$, which implies R is *GWS*. □

Since central symmetric rings are Abel and the ring in Example 2.2 is not Abel, the converse of Proposition 2.3 is not true, in general.

Proposition 2.4. *Let R be a GWS ring. If R satisfies one of the following conditions, then R is central symmetric*:

(1) $N(R) \subseteq Z(R)$;
(2) $J(R) \subseteq Z(R)$.

Proof. Let $abc = 0$. Then $ab(cy) = 0$ for any $y \in R$. Since R is *GWS*, $bacy \in N(R)$. Hence $bacR \subseteq N(R)$, one obtains that $bac \in N(R) \cap J(R)$. Thus (1) and (2) all implies $bac \in Z(R)$. □

Clearly, subrings of *GWS* rings are *GWS*. Especially, if R is a *GWS* ring and $e \in E(R)$, then eRe is *GWS*. But the converse is not true, in general.

Example 2.5. Let D be a field and $R = \begin{pmatrix} D & D \\ D & D \end{pmatrix}$. Since $E_{11}E_{21}E_{12} = 0$ and $E_{21}E_{11}E_{12} = E_{22} \notin N(R)$, R is not a *GWS* ring. But $E_{11} \in E(R)$ and $E_{11}RE_{11} \cong D$ is *GWS*.

Following [16], a ring R is called *quasi-normal* if $eR(1-e)Re = 0$ for any $e \in E(R)$. Clearly, Abel rings are quasi-normal. But the following Example 2.12 implies Abel rings need not be *GWS*. Hence quasi-normal rings need not be *GWS*.

Now let F be a field and $R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$. Then following Theorem 2.7 shows that R is a *GWS* ring. Let $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in E(R)$. Then $eR(1-e)Re = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, R is not quasi-normal. Hence *GWS* rings need not be quasi-normal.

Proposition 2.6. *Let R be a quasi-normal ring and $e \in E(R)$. If eRe and $(1-e)R(1-e)$ are all GWS, then R is GWS.*