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A 3-manifold group which is not four dimensional linear

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ABSTRACT

We give examples of closed orientable graph 3-manifolds having a fundamental group which is not a subgroup of $GL(4, \mathbb{F})$ for any field $\mathbb{F}$. This answers a question in the Kirby problem list from 1977 which is credited to the late William Thurston.

1. Introduction

As part of Thurston's revolutionary understanding of 3 dimensional geometry and topology, he established that many 3-manifolds have hyperbolic structures. The fundamental group of a hyperbolic 3-manifold is a subgroup of $PSL(2, \mathbb{C})$ that lifts to $SL(2, \mathbb{C})$, so is linear (which here will always mean a subgroup of $GL(n, \mathbb{F})$ for $\mathbb{F}$ a field) in 2 dimensions. Now $SL(2, \mathbb{C}) \leqslant GL(4, \mathbb{R})$ because a $\mathbb{C}$-linear map of $\mathbb{C}^2$ is an $\mathbb{R}$-linear map of $\mathbb{R}^4$, so we can ask whether every finitely generated 3-manifold group embeds in $GL(4, \mathbb{R})$. In fact this is Question 3.33 Part (A) in the first version of the Kirby problem list which dates from 1977 and is credited to Thurston, with Part (B) asking whether these groups are all residually finite. Now linearity of a finitely generated group implies residual finiteness and Part (B) was shown to be true on the acceptance of Perelman's solution to Geometrisation, because Thurston indicated and Hempel proved in [5] that residual finiteness is preserved when constructing 3-manifolds from their geometric pieces which themselves will have linear fundamental group. After this, Aschenbrenner and Friedl showed in [2] that finitely generated 3-manifold groups are virtually residually finite-p for all but finitely many primes p. This can be seen as further evidence that all of these groups are linear because this property is implied by linearity but is stronger than being residually finite.

Now the question of linearity is still open for 3-manifold groups. However a large amount of recent activity using the work [11] of Wise, such as [7,1] and [10], means it remains only to show linearity for those closed graph 3-manifolds which fail to possess a metric of nonpositive curvature, as then all compact prime 3-manifolds would have linear fundamental group and a free product of linear groups is linear.

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However, even if all 3-manifold groups are linear, we can still ask the stronger question: is there $n \in \mathbb{N}$ such that all finitely generated 3-manifold groups embed in $GL(n, \mathbb{R})$? In this paper we show that we cannot take $n = 4$, thus answering Thurston's Question 3.33 Part (A) in the negative. Indeed the answer is still negative even on replacing $\mathbb{R}$ with any field of any characteristic. We describe the 3-manifold in Section 2 and give details of its fundamental group but here we can summarise it thus: take two copies of the product of the punctured torus and the circle and form the graph manifold by identifying the boundary tori, with some conditions on the monodromy. The whole argument relies only on using Jordan normal form up to 4 by 4 matrices and considering the centraliser of a matrix in Jordan normal form. However the key idea is this: the circle in the first product 3-manifold requires a matrix having a big centraliser (by which we mean it contains a nonabelian free group). But as we do not allow ourselves to identify the two circles, this centraliser cannot contain the whole 3-manifold group. This argument applies also to the circle on the other side and if these two elements are both diagonalisable then they are simultaneously diagonalisable as they commute. This forces a block structure for all the matrices in each of the two pieces of the graph manifold and in Section 3 we show by an easy examination of the possible cases for both block decompositions in 4 dimensions that this cannot occur, because the diagonal entries of the circle elements will be roots of unity so these elements will have finite order.

In Section 4 we show how this conclusion that the eigenvalues are roots of unity generalises to arbitrary matrices over an algebraically closed field, by replacing simultaneous diagonalisation by simultaneous triangularisation. This then allows the positive characteristic case to be eliminated first by a quick argument, leaving only the field $\mathbb{C}$ without loss of generality. However in Section 5 we now have to deal with the circle elements having more complicated Jordan normal forms. Although their eigenvalues are still roots of unity, such matrices may of course have infinite order in the characteristic zero case. This section requires some rather more specialised arguments which we feel would not extend quickly to dimensions above 4, unlike those in the earlier sections. However the basis of these arguments is just taking each possible Jordan normal form for the circle elements and working out the centralisers.

In the last section we make a few related comments, including noting that our graph manifolds have already appeared in the literature where they were shown not to have a metric of nonpositive curvature (so the linearity of these 3-manifold groups is still open) and to be nonfibred but virtually fibred.

2. Description of the graph manifolds

We can form a closed orientable graph 3-manifold in the following way: let $S_{g,1}$ be the compact orientable surface of genus $g \geqslant 1$ with one boundary component. We know that $\pi_1(S_{g,1}) = F_{2g}$, the free group of rank $2g$, and we let $A \in F_{2g}$ be the element given by the boundary curve (oriented in some way). On forming the product manifold $M_1 = S_{g,1} \times S^1$ (which can be regarded as a trivial Seifert fibre space) we have that the group $G_1 = \pi_1(M_1)$ is isomorphic to $F_{2g} \times \mathbb{Z}$ with the element S generating $\mathbb{Z}$ being in the centre of this fundamental group. Moreover we have $\langle A, S\rangle = \mathbb{Z} \times \mathbb{Z}$ as this forms the fundamental group of the boundary torus ∂M_1.

We now take another manifold M_2 of this type with fundamental group G_2 (here the genus g' of our new surface $S_{g',1}$ does not need to equal g, although in Section 5 we will require that $g = g' = 1$) with B the corresponding peripheral element of $S_{g',1}$ and T the equivalent generator of the centre of $\pi_1(S_{g',1} \times S_1)$. Let M be the closed orientable graph manifold $M_1 \#_f M_2$ where $f : \partial M_1 \to \partial M_2$ is a homeomorphism of the torus which identifies the boundaries of the two 3-manifolds. This means that $\pi_1(M)$ is equal to the amalgamated free product $(F_{2g} \times \mathbb{Z}) *_\theta (F_{2g'} \times \mathbb{Z})$ where $\theta : \langle A, S\rangle \to \langle B, T\rangle$ is an isomorphism. The automorphisms of $\mathbb{Z} \times \mathbb{Z}$ are of course elements of $GL(2, \mathbb{Z})$ so we have integers i, j, k, l with $il - jk = \pm 1$ such that $B = A^i S^j$ and $T = A^k S^l$. Although it seems that the sign affects the orientability of M, we can assume that $il - jk = 1$ for $\pi_1(M)$ because we can replace T by T^{-1} (thus k and l by $-k$ and $-l$) without changing the group. For here on we do not consider 3-manifolds as we only need to examine the group