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Cellular categories

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ABSTRACT

We study locally presentable categories equipped with a cofibrantly generated weak factorization system. Our main result is that these categories are closed under 2-limits, in particular under pseudopullbacks. We give applications to deconstructible classes in Grothendieck categories. We discuss pseudopullbacks of combinatorial model categories.

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1. Introduction

We introduce cellular categories as categories equipped with a class of morphisms containing all isomorphisms and closed under pushout and transfinite composite (= transfinite composition). The special case is a category equipped with a weak factorization system, which includes categories equipped with a factorization system. The latter categories are called "structured" in [1]. Cellular categories are abundant in homotopy theory because any Quillen model category carries two weak factorization systems, i.e., two cellular structures given by cofibrations and trivial cofibrations, resp. There are also various concepts of "cofibration categories" equipped with cofibrations and weak equivalences (see [17] for a recent survey). One can do homotopy theory in any category equipped with a weak factorization system because we have cylinder objects and hence homotopies there (see [12]). Cellular category does not need to have weak factorizations – for example pure monomorphisms in certain locally finitely presentable categories (see [8]). However, in a locally presentable category, one always has weak factorizations whenever cellular morphisms are generated by a set of morphisms. The left part of the corresponding factorization system consists of retracts of cellular morphisms. In harmony with the J. Smith's concept of a combinatorial model category, we call such cellular categories combinatorial.

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Our main result is that combinatorial cellular categories are closed under constructions of limit type. Like for locally presentable (or accessible categories) these limits should be defined in the framework of 2-categories and they can be reduced to products, inserters and equifiers (see [15] and [3]). These limits are called PIE-limits. The consequence is that they include both lax limits and pseudolimits. It turns out that the key step is the closedness under pseudopullbacks and the key ingredience is the use of good colimits introduced by Lurie [13] and further developed by the first author in [14]. Lurie used good colimits for lifting cellular structure to functor categories, which is a limit type construction.

Our starting point is [16] and we are using notation from that paper. Among others, the present paper links combinatorial cellular categories with deconstructible classes of objects in Grothendieck abelian categories (see [5,9,10,19,20]) where good colimits are replaced by generalized Hill lemma. Our limit theorem for combinatorial cellular categories implies some limit theorems for deconstructible classes proved in [19] and [5].

2. Combinatorial categories

Let $\mathcal{X}$ be a class of morphisms in a category $\mathcal{K}$. Then $\mathrm{Po}(\mathcal{X})$ denotes the class of pushouts of morphisms in $\mathcal{X}$: $f \in \mathrm{Po}(\mathcal{X})$ iff f is an isomorphism or there is a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ X \arrow[u] \arrow[r, "g"'] & Y \arrow[u] \end{tikzcd}$$

with $g \in \mathcal{X}$.

$\mathrm{Tc}(\mathcal{X})$ denotes the class of transfinite composites (= compositions) of morphisms from $\mathcal{X}$: $f \in \mathrm{Tc}(\mathcal{X})$ iff there is a smooth chain $(f_{ij} : A_i \to A_j)_{i \leqslant j \leqslant \lambda}$ (i.e., λ is an ordinal, $(f_{ij} : A_i \to A_j)_{i<j}$ is a colimit for any limit ordinal $j \leqslant \lambda$) such that $f_{i,i+1} \in \mathcal{X}$ for each $i < \lambda$ and $f = f_{0\lambda}$.

$\mathrm{Rt}(\mathcal{X})$ denotes the class of retracts of morphisms in $\mathcal{X}$ in the category $\mathcal{K}^2$ of morphisms of $\mathcal{K}$.

Definition 2.1. A cocomplete category $\mathcal{K}$ is called *cellular* if it is equipped with a class $\mathcal{C}$ of morphisms closed under pushout and transfinite composite.

Morphisms belonging to $\mathcal{C}$ are called *cellular* and $\mathcal{C}$ will be often denoted as $\mathrm{cell}(\mathcal{K})$. Given a class $\mathcal{X}$ of morphisms of a cocomplete category $\mathcal{K}$ then $\mathrm{cell}(\mathcal{X})$ denotes the closure of $\mathcal{X}$ under pushout and transfinite composite. In fact, $\mathrm{cell}(\mathcal{X})$ consists of transfinite composites of pushouts of morphisms from $\mathcal{X}$. We say that the cellular category $(\mathcal{K}, \mathrm{cell}(\mathcal{X}))$ is *cellularly generated* by $\mathcal{X}$. In a cellular category $\mathcal{K}$, let

$$\mathrm{cof}(\mathcal{K}) = \mathrm{Rt}\,\mathrm{cell}(\mathcal{K}).$$

Elements of this class are called *cofibrations*.

Lemma 2.2. *Let $\mathcal{K}$ be a cellular category. Then $(\mathcal{K}, \mathrm{cof}(\mathcal{K}))$ is a cellular category.*

Proof. It is easy to see that $\mathrm{cof}(\mathcal{K})$ is closed under pushout. Let $f_0 : A_0 \to A_1$ and $f_1 : A_1 \to A_2$ be two composable cofibrations. Following [16] 2.1(5), there are cellular morphisms $g_0 : A_0 \to B_1$, $h_1 : A_1 \to C_2$ and morphisms $u_1 : A_1 \to B_1$, $r_1 : B_1 \to A_1$, $v_2 : A_2 \to C_2$, $s_2 : C_2 \to A_2$ such that u_1, r_1 make f_0 a retract of g_0 in $A_0\backslash\mathcal{K}$ and v_2, s_2 make f_1 a retract of h_1 in $A_1\backslash\mathcal{K}$. Consider a pushout

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