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Direct limits of Cohen–Macaulay rings

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ABSTRACT

Let A be a direct limit of a direct system of Cohen–Macaulay rings. In this paper, we describe the Cohen–Macaulay property of A. Our results indicate that A is not necessarily Cohen–Macaulay. We show A is Cohen–Macaulay under various assumptions. As an application, we study Cohen–Macaulayness of non-affine normal semigroup rings.

1. Introduction

Let $H \subseteq \mathbb{Z}^n$ be a *normal* and *affine* semigroup and let k be a field. A well-known result of Hochster says that $k[H]$ is Cohen–Macaulay, see [16, Theorem 1]. Suppose now that H is a non-affine normal semigroup. Then $k[H]$ is not Noetherian. Recently, the notion of Cohen–Macaulayness has been generalized to the non-Noetherian situation, see [15] and [2]. We intend to investigate the Cohen–Macaulayness of $k[H]$ in this context. In view of Example 2.3, $k[H]$ is a direct limit of Noetherian Cohen–Macaulay rings when $H \subseteq \mathbb{Z}^n$ is normal. This motivates us to ask:

Question 1.1. Is Cohen–Macaulayness closed under taking direct limit?

Our aim in this paper is to study the above question and its connection with Cohen–Macaulay properties of normal semigroup rings. Noetherian rings which are flat direct limit of a family of rings with certain properties was studied by Marot [17] and Doretti [11]. Also, the regular and complete intersection analogues of Question 1.1 follows by an immediate application of the theory of Andre–Quillen cohomology when the direct limit is Noetherian, see Remark 3.12.

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There are many candidates for definition of non-Noetherian Cohen–Macaulay rings, see [2, Definition 3.1]. To see relations between them, we recommend [2, 3.2 Relations]. All of them are the same, if the base ring is Noetherian. By using Zariski desingularization, we give a direct system of Noetherian regular rings $\{A_i: i \in I\}$ such that $\varinjlim_{i\in I} A_i$ is not Cohen–Macaulay in the sense of ideals, see Definition 3.2 and Example 3.6. In Section 3 we show that direct limit of Noetherian Cohen–Macaulay rings is Cohen–Macaulay in the sense of ideals in many situations, see Proposition 3.5 and Corollaries 3.7, 3.8 and 3.13 for its application. For the Gorenstein version of Question 1.1, see Corollary 3.11.

Considering [15], there is a description of the Cohen–Macaulay property for non-Noetherian rings. We call it Cohen–Macaulayness in the sense of Hamilton–Marley. In Section 4, we study the behavior of this property under taking direct limit, see Proposition 4.2. As an application, we give Cohen–Macaulayness of infinite dimensional Veronese and locally finitely generated normal semigroup rings in the sense of Hamilton–Marley, see Corollary 4.4 and Example 4.5. In Example 4.7, we give a direct system of Noetherian Cohen–Macaulay rings such that its direct limit is not Cohen–Macaulay in the sense of Hamilton–Marley. Let H be the normal semigroup $\{(a,b) \in \mathbb{N}_0^2 \mid 0 \leqslant b/a < \infty\} \cup \{(0,0)\}$ and let k be a field. Note that H is a standard example of a non-affine semigroup. As an application of Example 4.7, in Theorem 4.10, we show

(i) $k[H]$ is not Cohen–Macaulay in the sense of ideals.
(ii) $k[H]$ is not weak Bourbaki unmixed.
(iii) $k[H]$ is Cohen–Macaulay in the sense of Hamilton–Marley.

Throughout this paper, rings are commutative (not necessarily Noetherian). Most of the results concerning direct limits in this paper require the direct system to be filtered. In the sequel all direct systems are assumed to be filtered.

2. Direct limits of Cohen–Macaulay rings: Examples

In this section we construct some examples of rings that are direct limits of Noetherian Cohen–Macaulay rings. In what follows, we will use all of them. Also, we study their Cohen–Macaulay properties in Examples 3.6, 4.7 and Theorem 4.10.

Definition 2.1. Let C be a submonoid of $\mathbb{Z}^n$ for some n and let k be a field.

(i) We write $k[C]$ for the vector space $k^{(C)}$, and denote the basis element of $k[C]$ which corresponds to $c \in C$ by X^c. This monomial notation is suggested by the fact that $k[C]$ carries a natural multiplication whose table is given by $X^c X^{c'} = X^{c+c'}$.
(ii) Recall that $C \subseteq \mathbb{Z}^n$ is said to be *normal* if, whenever $m(c-c') \in C$ for some positive integer m and $c, c' \in C$, then $c - c' \in C$.
(iii) Let N be a submonoid of a commutative monoid M. The *integral closure* of N in M is the submonoid

$$\widehat{N}_M := \{x \in M \mid mx \in N \text{ for some positive integer } m \in \mathbb{N}\}$$

of M. One calls N integrally closed in M, if $N = \widehat{N}_M$.

Lemma 2.2. *Let C be a normal submonoid of $\mathbb{Z}^n$. Then there is a direct system $(C_\gamma, f_{\gamma\delta})$ of finitely generated normal submonoids of C such that $C = \varinjlim_{\gamma\in\Gamma} C_\gamma$, where $f_{\gamma\delta} : C_\gamma \to C_\delta$ is the inclusion map for $\gamma, \delta \in \Gamma$ with $\gamma \leqslant \delta$.*

Proof. Look at the set $\Gamma := \{X \subseteq C \mid X \text{ is finite}\}$. We direct Γ by means of inclusion. Set $C_X := \mathbb{Z}^{\geqslant 0} X$ and $C'_X := \widehat{(C_X)}_{\mathbb{Z}C_X}$ for all $X \in \Gamma$. One sees easily that C'_X is normal. In view of [8, Proposition 2.22], C'_X