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On real one-sided ideals in a free algebra

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ABSTRACT

In real algebraic geometry there are several notions of the radical of an ideal I. There is the vanishing radical $\sqrt{I}$ defined as the set of all real polynomials vanishing on the real zero set of I, and the real radical $\sqrt[\mathrm{re}]{I}$ defined as the smallest real ideal containing I. (Neither of them is to be confused with the usual radical from commutative algebra.) By the real Nullstellensatz, $\sqrt{I} = \sqrt[\mathrm{re}]{I}$. This paper focuses on extensions of these to the free algebra $\mathbb{R}\langle x, x^*\rangle$ of noncommutative real polynomials in $x = (x_1, \ldots, x_g)$ and $x^* = (x_1^*, \ldots, x_g^*)$.

We work with a natural notion of the (noncommutative real) zero set $V(I)$ of a left ideal I in $\mathbb{R}\langle x, x^*\rangle$. The vanishing radical $\sqrt{I}$ of I is the set of all $p \in \mathbb{R}\langle x, x^*\rangle$ which vanish on $V(I)$. The earlier paper (Cimprič et al. [6]) [6] gives an appropriate notion of $\sqrt[\mathrm{re}]{I}$ and proves $\sqrt{I} = \sqrt[\mathrm{re}]{I}$ when I is a finitely generated left ideal, a free $*$-Nullstellensatz. However, this does not tell us for a particular ideal I whether or not $I = \sqrt[\mathrm{re}]{I}$, and that is the topic of this paper. We give a complete solution for monomial ideals and homogeneous principal ideals. We also present the case of principal univariate ideals with a degree two generator and find that it is very messy. We discuss an algorithm to determine if $I = \sqrt[\mathrm{re}]{I}$ (implemented under NCAlgebra) with finite run times and provable effectiveness.

1. Introduction

The introduction begins with definitions and a little motivation for them. Then it sketches the main results of this paper together with links to where they are found.

1.1. Zero sets in free algebras

Let $\langle x, x^*\rangle$ be the monoid freely generated by $x = (x_1, \ldots, x_g)$ and $x^* = (x_1^*, \ldots, x_g^*)$, i.e., $\langle x, x^*\rangle$ consists of **words** in the $2g$ noncommuting letters $x_1, \ldots, x_g, x_1^*, \ldots, x_g^*$ (including the empty word $\varnothing$ which plays the role of the identity 1). Let $\mathbb{R}\langle x, x^*\rangle$ denote the $\mathbb{R}$-algebra freely generated by x, x^*, i.e., the elements of $\mathbb{R}\langle x, x^*\rangle$ are **polynomials** in the noncommuting variables x, x^* with coefficients in $\mathbb{R}$. Equivalently, $\mathbb{R}\langle x, x^*\rangle$ is the **free $*$-algebra** on x. The length of the longest word in a noncommutative polynomial $f \in \mathbb{R}\langle x, x^*\rangle$ is the **degree** of f and is denoted by $\deg(f)$. The set of all words of degree at most k is $\langle x, x^*\rangle_k$, and $\mathbb{R}\langle x, x^*\rangle_k$ is the vector space of all noncommutative polynomials of degree at most k.

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Given a g-tuple $X = (X_1, \ldots, X_g)$ of same size square matrices over $\mathbb{R}$, write $p(X)$ for the natural evaluation of p at X. For $S \subseteq \mathbb{R}\langle x, x^*\rangle$ we introduce

$$V(S)^{(n)} = \left\{(X, v) \in (\mathbb{R}^{n\times n})^g \times \mathbb{R}^n \mid p(X)v = 0 \text{ for every } p \in S\right\},$$

and define the **zero set** of S to be

$$V(S) = \bigcup_{n\in\mathbb{N}} V(S)^{(n)} = \{(X, v) \mid p(X)v = 0 \text{ for every } p \in S\}.$$

To each subset T of $\bigcup_{n\in\mathbb{N}} \left((\mathbb{R}^{n\times n})^g \times \mathbb{R}^n\right)$ we associate the **left ideal**

$$\mathcal{I}(T) = \{p \in \mathbb{R}\langle x, x^*\rangle \mid p(X)v = 0 \text{ for every } (X, v) \in T\}.$$

For a left ideal I of $\mathbb{R}\langle x, x^*\rangle$, we call

$$\sqrt{I} := \mathcal{I}\big(V(I)\big)$$

the **vanishing radical** of I.[1] Evidently $\sqrt{I}$ is a left ideal. We say that I **has the left Nullstellensatz property** if $\sqrt{I} = I$. Now we describe a class of ideals which has this property.

A polynomial p is **analytic** if it has no transpose variables, that is, no x_i^*. For example, $p(x) = 1 + x_1x_2 + x_1^3 + x_2^5$ is analytic while $p(x) = 1 + x_1^*x_2 + (x_1^*)^3 + x_2^5$ is not analytic. There is a strong Nullstellensatz for left ideals generated by analytic polynomials in [15]. This strengthens an earlier result proved by Bergman [13]; see also [3] for a survey on noncommutative Nullstellensätze.

Theorem 1.1. *Let $p_1, \ldots, p_m$ be analytic polynomials, and let I be the left ideal generated by those p_i. Then $\sqrt{I} = I$, i.e.,*

$$\big(\forall j\ p_j(X)v = 0\big) \Longrightarrow q(X)v = 0 \quad \text{iff} \quad q = f_1p_1 + \cdots + f_mp_m.$$

We pause to make two remarks related to Theorem 1.1. First, no powers q^k are needed, contrary to the case in the classical commutative Hilbert Nullstellensatz or the real Nullstellensatz. This absence of powers seems to be the pattern for free algebra situations.

Second, while there are other notions of zero of a free polynomial, the one used here is particularly suited for studying left ideals and has proved fruitful in a variety of other contexts; e.g. [15,14]. One alternative notion is to say that X is a zero of p if $p(X) = 0$. However, in this case, for $R \subseteq \bigcup_{n\in\mathbb{N}}(\mathbb{R}^{n\times n})^g$, the set $\{p \mid p(X) = 0 \text{ for all } X \in R\}$ is a two-sided ideal. Another choice is to declare X a zero of p if $p(X)$ fails to be invertible, but then $\{p \mid \det(p(X)) = 0 \text{ for all } X \in R\}$ is not closed under sums.

What about ideals generated by p_j which are not analytic? To shed light on the basic question of which ideals have the left Nullstellensatz property, we seek an algebraic description of the vanishing radical $\sqrt{I}$ similar to the notion of real radical in the classical real algebraic geometry; cf. [2, Chapter 4], [23, Chapter 2], [27, Chapter 4] or [30]. For this we introduced real ideals in [6]. Now we recall these definitions.

A left ideal I of $\mathbb{R}\langle x, x^*\rangle$ is said to be **real** if for every $a_1, \ldots, a_r \in \mathbb{R}\langle x, x^*\rangle$ such that

$$\sum_{i=1}^{r} a_i^*a_i \in I + I^*,$$

we have that $a_1, \ldots, a_r \in I$. Here I^* is the right ideal $I^* = \{a^* \mid a \in I\}$. An intersection of a family of real ideals is a real ideal. For a left ideal J of $\mathbb{R}\langle x, x^*\rangle$ we call the ideal

$$\sqrt[\text{re}]{J} = \bigcap_{\substack{I\supseteq J\\ I \text{ real}}} I = \text{the smallest real ideal containing } J$$

the real radical of J. It is not hard to show for any left ideal I that

$$I \subseteq \sqrt[\text{re}]{I} \subseteq \sqrt{I};$$

see [6]. The main result of [6] is a real Nullstellensatz which states as follows.

Theorem 1.2 ([6, Theorem 1.6]). *A finitely generated left ideal I in $\mathbb{R}\langle x, x^*\rangle$ satisfies $\sqrt[\text{re}]{I} = \sqrt{I}$. Thus I has the left Nullstellensatz property if and only if it is real.*

This result is not true for infinitely generated ideals, as shown in Example 2.2.

A quantitative version of this theorem gives bounds (which we shall need) on the degrees of the polynomials involved.

Theorem 1.3 ([6, Theorem 2.5]). *Let I be a left ideal in $\mathbb{R}\langle x, x^*\rangle$ generated by polynomials of degree bounded by d. Then I is real if and only if whenever $q_1, \ldots, q_k$ are polynomials with $\deg(q_j) < d$ for each j, and $\sum_{i=1}^{\ell} q_i^*q_i \in I + I^*$, then $q_j \in I$ for each j.*

These results give a clean equivalence but do not tell us whether or not a particular ideal has the left Nullstellensatz property. This paper focuses on examples of ideals I for which we can determine if $I = \sqrt{I}$.

[1] In [6] this radical was denoted $\sqrt[\Pi]{\ }$. Since in this article only radicals with respect to finite dimensional representations are considered, the Π has been dropped.