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The Green rings of pointed tensor categories of finite type

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ABSTRACT

In this paper, we compute the Clebsch–Gordan formulae and the Green rings of connected pointed tensor categories of finite type.

1. Introduction

Throughout the paper, k is an algebraically closed field of characteristic zero, and (co)algebras, (co)modules, categories, etc. are over k. Concepts and notations about tensor categories are adopted from [7].

Let $\mathcal{C}$ be a tensor category, that is, a locally finite abelian rigid monoidal category in which the neutral object is simple. Note that the Jordan–Hölder theorem and the Krull–Schmidt theorem hold in $\mathcal{C}$. For each $M \in \mathcal{C}$, let $[M]$ denote the associated iso-class. The Green ring $\mathcal{GR}(\mathcal{C})$ of $\mathcal{C}$ is the ring with generators the iso-classes $[M]$ of $\mathcal{C}$, and relations

$$[M] + [N] = [M \oplus N], \qquad [M] \cdot [N] = [M \otimes N].$$

By the Krull–Schmidt theorem, the ring $\mathcal{GR}(\mathcal{C})$ is a free $\mathbb{Z}$-module, and the set of indecomposable iso-classes of $\mathcal{C}$ forms a basis. When $\mathcal{C} = H$-mod, where H is a Hopf algebra, the ring $\mathcal{GR}(\mathcal{C})$ is also called the Green ring of H.

The Green ring $\mathcal{GR}(\mathcal{C})$ provides a convenient way of organizing information about direct sums and tensor products of $\mathcal{C}$. When $\mathcal{C}$ is the category of modular representations of a finite group, the ring $\mathcal{GR}(\mathcal{C})$ was investigated systematically for the first time by J.A. Green [9], hence the notion Green ring is widely used in the literature. Green rings have played an important role in the modular representation theory; see [2] and references therein.

Without a doubt, Green rings should be equally important in the study of those tensor categories of (co)modules over (co)quasi-Hopf algebras (or quasi-quantum groups) which are not semisimple; see [7,8]. However, so far there are not many results obtained about the Green rings of such tensor categories due to the obvious complexity. In [16], the Green ring (called the representation ring there) of the quantum double of a finite group was described by a direct sum decomposition. In a subsequent work [17], the Green ring of the twisted quantum double of a finite group was considered and some results about the Grothendieck ring, that is the quotient of the Green ring by the ideal of short exact sequences, were obtained. Recently, the Green rings of the Taft algebras and the generalized Taft algebras were presented by generators and relations in [3,14] respectively.

The aim of this paper is to compute the Clebsch–Gordan formulae and to present the Green rings of pointed tensor categories of finite type as classified in [13]. Such tensor categories are actually the comodule categories of some pointed

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coquasi-Hopf algebras of finite corepresentation type, or equivalently the module categories of some elementary quasi-Hopf algebras of finite representation type, in which there are only finitely many iso-classes of indecomposable objects, and as such their Green rings are suitable to study. Among them, the module categories of the Taft and generalized Taft algebras are particular examples. In addition, these tensor categories can be presented by quiver representations and we can take full advantage of the handy quiver techniques (see e.g. [1,18]) for the computations. In Section 2, we recall some necessary facts about pointed tensor categories of finite type. The Clebsch–Gordan formulae for such tensor categories are computed in Section 3. Finally in Section 4, the Green rings are determined. The results obtained here extend those in [5,3,14] to a much greater range in a unified way.

2. Pointed tensor categories of finite type

In this section, we recall pointed tensor categories of finite type and their presentations via quiver representations.

2.1. Let $n > 1$ be a positive integer and $C_n = \langle g | g^n = 1\rangle$ the cyclic group of order n. By Z_n we denote the cyclic quiver with vertices indexed by C_n and with arrows $a_i : g^i \to g^{i+1}$ where the indices i are understood as integers modulo n. Let p_i^l denote the path $a_{i+l-1} \cdots a_{i+1} a_i$ of length l. The k-span of $\{p_i^l \mid 0 \le i \le n-1,\ l \ge 0\}$ is called the associated path space of the quiver Z_n, and denoted by kZ_n.

From now on, let m' denote the remainder of the division of the integer m by n. There is a natural coalgebra structure on kZ_n with coproduct and counit given by

$$\Delta(p_i^l) = \sum_{m=0}^{l} p_{(i+m)'}^{l-m} \otimes p_i^m, \qquad \varepsilon(p_i^l) = \delta_{l,0} = \begin{cases} 1, & l = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

This is the so-called path coalgebra of the quiver Z_n.

2.2. The quiver Z_n is in fact a Hopf quiver in the sense of [6]. By [10], there exist graded coquasi-Hopf algebra, also known as Majid algebra, structures on the path coalgebra kZ_n. Moreover, the graded coquasi-Hopf structures are parameterized by some set of 1-dimensional projective representation of C_n; see [11] for a general theorem.

Explicit graded coquasi-Hopf structures on kZ_n are given in [13]. To state this result, first we need to fix some notations. For any $\hbar \in k$, define $m_\hbar = 1 + \hbar + \cdots + \hbar^{m-1}$ and $m!_\hbar = 1_\hbar \cdots m_\hbar$. The Gaussian binomial coefficient is defined by $\binom{m+n}{m}_\hbar := \frac{(m+n)!_\hbar}{m!_\hbar n!_\hbar}$. Let $\mathbb{q}$ be a primitive n-th root of unity. Assume $0 \le s \le n-1$ is an integer and q an n-th root of $\mathbb{q}^s$. For each pair (s, q), there is a unique graded coquasi-Hopf algebra $kZ_n(s, q)$ on kZ_n with multiplication given by

$$p_i^l \cdot p_j^m = \mathbb{q}^{-sjl} q^{-jl} \mathbb{q}^{\frac{s(i+l')[m+j-(m+j)']}{n}} \binom{l+m}{l}_{\mathbb{q}^{-s} q^{-1}} p_{(i+j)'}^{l+m}. \tag{2.2}$$

Let d be the order of q. Clearly, $d | n$ if $s = 0$ and $d = \frac{n^2}{(s,n^2)}$ if $1 \le s \le n-1$. Let $kZ_n(d)$ denote the subcoalgebra of kZ_n which as a k-space is spanned by $\{p_i^l \mid 0 \le i \le n-1,\ 0 \le l \le d-1\}$. The multiplication (2.1) is closed inside $kZ_n(d)$ and it becomes a graded coquasi-Hopf algebra, denoted by $M(n, s, q)$.

2.3. A pointed tensor category of finite type is a tensor category in which there are only finitely many iso-classes of indecomposable objects and whose simple objects are invertible; see [7,13].

By $\mathcal{C}(n, s, q)$ we denote the category of finite-dimensional right $M(n, s, q)$-comodules. It is proved in [13] that any pointed tensor category of finite type consists of finitely many identical components, and the connected component containing the neutral object is equivalent to a deformation of some $\mathcal{C}(n, s, q)$, and see [13, Theorem 4.2 and Corollary 4.4] for a full description.

In this paper we focus on the Clebsch–Gordan formulae and the Green ring of $\mathcal{C}(n, s, q)$. The results can be easily extended to the non-connected case. Now we give an explicit presentation of $\mathcal{C}(n, s, q)$ by quiver representations. Recall that a representation of the quiver Z_n is a collection $V = (V_i, T_i)_{0 \le i \le n-1}$ where V_i is a vector space corresponding to the vertex g^i and $T_i : V_i \to V_{i+1}$ is a linear map corresponding to the arrow a_i. Given a path p_i^l, we define a corresponding linear map T_i^l as follows. If $l = 0$, then put $T_i^0 = \mathrm{Id}_{V_i}$. If $l > 0$, put $T_i^l = T_{i+l-1} \cdots T_{i+1} T_i$. The category $\mathcal{C}(n, s, q)$ consists of the representations V of Z_n such that $T_i^l = 0$ whenever $l \ge d$.

The indecomposable objects of $\mathcal{C}(n, s, q)$ can be described as follows. Assume $0 \le i \le n-1$ and $0 \le e \le d-1$. Let $V(i, e)$ be a vector space of dimension $e+1$ with a basis $\{v_m^i\}_{0 \le m \le e}$. $V(i, e)$ is made into a representation of Z_n by putting $V(i, e)_j$ the k-span of $\{v_m^i | (i+m)' = j\}$ and letting $T(i, e)_j$ maps v_m^i to v_{m+1}^i if $(i+m)' = j$. Here the vector v_{e+1}^i is understood as 0 by convention. Clearly $V(i, e)$ is an object in $\mathcal{C}(n, s, q)$ and $\{V(i, e)\}_{0 \le i \le n-1,\ 0 \le e \le d-1}$ is a complete set of iso-classes of indecomposable objects of $\mathcal{C}(n, s, q)$.

2.4. For later computations, we need to know how the quiver representation $V(i, e)$ is viewed as a right $M(n, s, q)$-comodule. Let

$$\delta : V(i, e) \to V(i, e) \otimes M(n, s, q)$$