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The homotopy theory of bialgebras over pairs of operads

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ABSTRACT

We endow the category of bialgebras over a pair of operads in distribution with a cofibrantly generated model category structure. We work in the category of chain complexes over a field of characteristic zero. We split our construction in two steps. In the first step, we equip coalgebras over an operad with a cofibrantly generated model category structure. In the second step we use the adjunction between bialgebras and coalgebras via the free algebra functor. This result allows us to do classical homotopical algebra in various categories such as associative bialgebras, Lie bialgebras or Poisson bialgebras in chain complexes.

0. Introduction

The goal of this paper is to define a model category structure for the categories of bialgebras governed by operads in distribution. The work of Drinfeld on quantum groups (see [2] and [3]) has initiated the study of bialgebra structures where the product and the coproduct belong to various types of algebras. Besides the classical Hopf algebras, examples include their non-commutative non-cocommutative variant, Lie bialgebras, and Poisson bialgebras. Applications range from knot theory in topology to integrable systems in mathematical physics. The theory of operads in distribution, introduced by Fox and Markl in [5], provides a convenient generalization of the classical categories of bialgebras defined by products and coproducts in distribution. The general idea is that there is an operad encoding the operations (where we have several inputs and a single output) and another operad encoding the cooperations (a single input and several outputs). The distributive law then formalizes the interplay between these operads, i.e. the compatibilities between operations and cooperations. We refer the reader to [14] for a detailed survey providing many examples of these generalized bialgebras. One may then wonder how to transpose homotopical algebra methods in this setting, as it has been done successfully for algebras over operads. The aim of this paper is precisely to define a closed model category structure for bialgebras over operads in distribution.

The existence of a cofibrantly generated model category structure on algebras over a suitable operad is a classical result, see [11]. When working over a field of characteristic zero, such a structure exists for any operad. Let $Ch_{\mathbb{K}}^{+}$ be the category of positively graded chain complexes. We denote by ${}^{P}Ch_{\mathbb{K}}^{+}$ the category of P-coalgebras in $Ch_{\mathbb{K}}^{+}$. To simplify, we only consider operads in the category of $\mathbb{K}$-vector spaces $Vect_{\mathbb{K}}$ in this paper. We use that $Ch_{\mathbb{K}}^{+}$ is tensored over $Vect_{\mathbb{K}}$ (in a way compatible with respect to internal tensor structures) to give a sense to the notion of P-coalgebra in this category $Ch_{\mathbb{K}}^{+}$. We use similar conventions when we deal with algebras over operads. In a first step we establish the existence of a cofibrantly generated model category structure for coalgebras over an operad:

Theorem 0.1. *Let P be an operad in $Vect_{\mathbb{K}}$ such that $P(0)=0$, $P(1)=\mathbb{K}$, and the vector spaces $P(n)$, $n>1$, are finite dimensional. The category of P-coalgebras ${}^{P}Ch_{\mathbb{K}}^{+}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^{P}Ch_{\mathbb{K}}^{+}$ is*

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(i) *a weak equivalence if* $U(f)$ *is a weak equivalence in* $Ch_{\mathbb{K}}^{+}$;
(ii) *a cofibration if* $U(f)$ *is a cofibration in* $Ch_{\mathbb{K}}^{+}$;
(iii) *a fibration if* f *has the right lifting property with respect to acyclic cofibrations.*

Note that an analogous result has been proven in [18] in the context of unbounded chain complexes. We follow another simpler approach. We do not address the same level of generality, but we obtain a stronger result. To be more precise, in contrast with [18], we obtain a cofibrantly generated structure. These generating cofibrations are crucial to transfer the model structure on bialgebras. Moreover, we do not need the hypothesis considered in [18] about the underlying operad (see [18, condition 4.3]). Our method is close to the ideas of [9]. Such a result also appears in [1], but for coalgebras over a quasi-free cooperad.

Then we transfer this cofibrantly generated model structure to the category of (P, Q)-bialgebras. We denote by ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ the category of (P, Q)-bialgebras in $Ch_{\mathbb{K}}^{+}$, where P encodes the operations and Q the cooperations. We use an adjunction

$$P : {}^{Q}Ch_{\mathbb{K}}^{+} \rightleftarrows {}_{P}^{Q}Ch_{\mathbb{K}}^{+} : U$$

to perform the transfer of model structure and obtain our main theorem:

Theorem 0.2. *The category of* (P, Q)*-bialgebras* ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ *inherits a cofibrantly generated model category structure such that a morphism* f *of* ${}_{P}^{Q}Ch_{\mathbb{K}}^{+}$ *is*

(i) *a weak equivalence if* $U(f)$ *is a weak equivalence in* ${}^{Q}Ch_{\mathbb{K}}^{+}$ *(i.e. a weak equivalence in* $Ch_{\mathbb{K}}^{+}$ *by definition of the model structure on* ${}^{Q}Ch_{\mathbb{K}}^{+}$*);*
(ii) *a fibration if* $U(f)$ *is a fibration in* ${}^{Q}Ch_{\mathbb{K}}^{+}$;
(iii) *a cofibration if* f *has the left lifting property with respect to acyclic fibrations.*

The strategy of the proofs is the following. First we prove Theorem 0.1. For this aim we construct, for any P-coalgebra A, a cooperad $U_{P^*}(A)$ called its enveloping cooperad, whose coalgebras are the P-coalgebras over A. It expresses the coproduct of A with a cofree coalgebra in terms of the evaluation of the associated enveloping cooperad functor. Axioms M2 and M3 are obvious. Axiom M1 is proved in an analogue way as in the case of algebras. The main difficulty lies in the proofs of M4 and M5. We use proofs inspired from that of [9] and adapted to our operadic setting. The enveloping cooperad plays a key role here. In order to produce the desired factorization axioms, our trick here is to use a slightly modified version of the usual small object argument. We use smallness with respect to injections systems running over a certain ordinal.

Then we prove our main result, Theorem 0.2. Axioms M1, M2 and M3 are easily checked, and axiom M4 follows from arguments similar to the case of coalgebras. The main difficulty is the proof of M5. We use mainly the small object argument for smallness with respect to injections systems, combined with a result about cofibrations in algebras over an operad due to Hinich [11].

Let us point out that in both cases, we cannot use the usual simplifying hypothesis of smallness with respect to all morphisms. This is due to the fact that Lemma 2.16, giving an essential smallness property for coalgebras, is only true for smallness with respect to injection systems.

Organization: the overall setting is reviewed in Section 1. We suppose some prerequisites concerning operads (see [15]) and give definitions of algebras and coalgebras over an operad. Then we define distributive laws from the monadic viewpoint, following [5]. Examples of monads and comonads include operads and cooperads. We recall some basic facts about the small object argument in cofibrantly generated model categories, in order to fix useful notations for the following.

The heart of this paper consists of Sections 2 and 3, devoted to the proofs of Theorems 0.1 and 0.2. The proof of Theorem 0.1 heavily relies on the notion of enveloping cooperad, which is defined in 2.2. In 2.3, we follow the argument line of [9], checking carefully where modifications are needed to work at our level of generality. Theorem 0.2 is proved in Section 3, by using adjunction properties to transfer the model structure obtained in Theorem 0.1. The crux here is a small object argument with respect to systems of injections of coalgebras.

1. Recollections

In this section, we first list some notions and facts about operads and algebras over operads. Then we review the interplay between monads and comonads by means of distributive laws and make the link with operads. It leads us to the crucial definition of bialgebras over pairs of operads in distribution. Finally, we recall a classical tool of homotopical algebra, namely the small object argument, aimed to produce factorizations in model categories. The material of this section is taken from [15,5] and [13].

1.1. Algebras and coalgebras over operads

For the moment, we work in the category of non-negatively graded chain complexes $Ch_{\mathbb{K}}$, where $\mathbb{K}$ is a field of characteristic zero (but we still assume that our operads belong to $Vect_{\mathbb{K}}$). We adopt most conventions of [15] and freely use the