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# Projective modules over overrings of polynomial rings and a question of Quillen

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ABSTRACT

Let $(R, \mathfrak{m}, K)$ be a regular local ring containing a field $k$ such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geqslant 1$. Let $g_1, \ldots, g_t$ be regular parameters of $R$ which are linearly independent modulo $\mathfrak{m}^2$. Let $A = R_{g_1 \cdots g_t}[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$, where $f_i(T) \in k[T]$ and $l_i = a_{i1}Y_1 + \cdots + a_{im}Y_m$ with $(a_{i1}, \ldots, a_{im}) \in k^m - (0)$. Then every projective $A$-module of rank $\geqslant t$ is free. Laurent polynomial case $f_i(l_i) = Y_i$ of this result is due to Popescu.

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## 1. Introduction

*In this paper, we will assume that rings are commutative Noetherian, modules are finitely generated, projective modules are of constant rank and $k$ will denote a field.*

Let $R$ be a ring and $P$ a projective $R$-module. We say that $P$ is *cancellative* if $P \oplus R^m \xrightarrow{\sim} Q \oplus R^m$ for some projective $R$-module $Q$ implies $P \xrightarrow{\sim} Q$. For simplicity of notations, we begin with a definition.

**Definition 1.1.** A ring $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ is said to be **of type** $R[d, m, n]$ if $R$ is a ring of dimension $d$, $Y_1, \ldots, Y_m$ are variables over $R$, each $f_i(T) \in R[T]$ and either each $l_i = Y_{i_j}$ for some $i_j$, or $R$ contains a field $k$ and $l_i = \sum_{j=1}^{m} a_{ij} Y_j - b_i$ with $b_i \in R$ and $(a_{i1}, \ldots, a_{im}) \in k^m - (0)$.

Let $A$ be a ring of the type $R[d, m, n]$. We say that $A$ is **of type** $R[d, m, n]^*$ if $f_i(T) \in k[T]$ and $b_i \in k$ for all $i$.

Let $A = R[Y_1, \ldots, Y_m, f_1(Y_1)^{-1}, \ldots, f_n(Y_n)^{-1}]$ be a ring of type $R[d, m, n]$ with $n \leqslant m$ and $l_i = Y_i$. If $P$ is a projective $A$-module of rank $\geqslant \max\{2, d+1\}$, then Dhorajia and Keshari [5, Theorem 3.12], proved that $E(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$ and hence $P$ is cancellative. This result was proved by Bass [2] in case $n = m = 0$; Plumstead [12] in case $m = 1$, $n = 0$; Rao [16] in case $n = 0$; Lindel [8] in case $f_i = Y_i$. Gabber [6] proved the following result: *Let $k$ be a field and $A$ a ring of type $k[0, m, n]$. Then every projective $A$-module is free*. We prove the following result (Theorema 3.4) which generalizes [5, Theorem 3.12] and is motivated by Gabber's result.

**Theorem 1.2.** *Let $A = R[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ be a ring of type $R[d, m, n]$ and $P$ a projective $A$-module of rank $\geqslant \max\{2, d+1\}$. Then $E(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$. In particular, $P$ is cancellative.*

The Bass–Quillen conjecture [3,15] says: *If $R$ is a regular ring, then every projective module over $R[X_1, \ldots, X_r]$ is extended from $R$*. In B–Q conjecture, we may assume that $R$ is a regular local ring, due to Quillen's local-global principal [15]: *For a ring $B$, projective module $P$ over $B[X_1, \ldots, X_r]$ is extended from $B$ if and only if $P_{\mathfrak{m}}$ is free for every maximal ideal $\mathfrak{m}$ of $B$*. We remark

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that Quillen's local global principal is also true for projective modules over positive graded rings [19, Theorem 3.1], whereas it is not true for Laurent polynomial rings [4, Example 2, p. 809].

Lindel [9] gave an affirmative answer to B–Q conjecture when $R$ is a *regular k-spot*, i.e. $R = R'_{\mathfrak{p}}$, where $R'$ is some affine $k$-algebra and $\mathfrak{p}$ is a regular prime ideal of $R'$. Using Lindel's result, Popescu [13] proved B–Q conjecture when $R$ is any regular local ring containing a field $k$.

Let $(R, \mathfrak{m})$ be a regular local ring. We say that $f \in \mathfrak{m}$ is a *regular parameter* of $R$ if $f$ is part of a minimal generating set of $\mathfrak{m}$. This is equivalent to $f \in \mathfrak{m} - \mathfrak{m}^2$. Further, let $g_1, \ldots, g_t \in \mathfrak{m}$ be regular parameters. Then $g_1, \ldots, g_t$ are linearly independent modulo $\mathfrak{m}^2$ if and only if $g_1, \ldots, g_t$ are part of a minimal generating set of $\mathfrak{m}$.

Quillen [15] had asked the following question whose affirmative answer would imply that B–Q conjecture is true: *Assume $(R, \mathfrak{m})$ is a regular local ring and $f \in \mathfrak{m}$ a regular parameter of $R$. Is every projective $R_f$-module free?*

Bhatwadekar and Rao [4] answered Quillen's question when $R$ is a regular $k$-spot. More generally, they proved: *Let $(R, \mathfrak{m})$ be a regular k-spot with infinite residue field and $f$ a regular parameter of $R$. If $B$ is one of $R$, $R(T)$ or $R_f$, then projective modules over $B[X_1, \ldots, X_r, Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}]$ are free.*

Rao [17] generalized above result as follows: *Let $(R, \mathfrak{m})$ be a regular k-spot with infinite residue field. Let $g_1, \ldots, g_t$ be regular parameters of $R$ which are linearly independent modulo $\mathfrak{m}^2$. If $A = R_{g_1 \ldots g_t}[X_1, \ldots, X_r, Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}]$, then projective $A$-modules of rank $\geqslant$ min $\{t, d/2\}$ are free.*

Popescu [14] generalized Rao's result as follows: *Let $(R, \mathfrak{m}, K)$ be a regular local ring containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geqslant 1$. Let $g_1, \ldots, g_t$ be regular parameters of $R$ which are linearly independent modulo $\mathfrak{m}^2$. If $A = R_{g_1 \ldots g_t}[X_1, \ldots, X_r, Y_1^{\pm 1}, \ldots, Y_s^{\pm 1}]$, then projective $A$-modules of rank $\geqslant t$ are free.*

We generalize Popescu's result as follows (Theorem 5.8):

**Theorem 1.3.** *Let $(R, \mathfrak{m}, K)$ be a regular local ring containing a field k such that either char $k = 0$ or char $k = p$ and tr-deg $K/\mathbb{F}_p \geqslant 1$. Let $g_1, \ldots, g_t$ be regular parameters of $R$ which are linearly independent modulo $\mathfrak{m}^2$. If $A = R_{g_1 \ldots g_t}[Y_1, \ldots, Y_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}]$ is a ring of type $R_{g_1 \ldots g_t}[d-1, m, n]^*$, then every projective $A$-module of rank $\geqslant t$ is free.*

Note that we can not expect (1.3) for rings of type $R[d, m, n]$. For example, let $R$ be either $\mathbb{R}[X, Y]_{(X,Y)}$ or $\mathbb{R}[[X, Y]]$ and $A = R[Z, f(Z)^{-1}]$ a ring of type $R[2, 1, 1]$, where $f(T) = T^2 + X^2 + Y^2$. Then stably free $A$-module $P$ of rank 2 given by the kernel of the surjection $(X, Y, Z) : A^3 \to A$ is not free. This will follow from the fact that $P$ over the rings $\mathbb{R}[X, Y, Z]_{(X,Y,Z)}[f(Z)^{-1}]$ or $\mathbb{R}[[X, Y, Z]][f(Z)^{-1}]$ is not free [4, p. 808] and [11, p. 366].

## 2. Preliminaries

Let $A$ be a ring and $M$ an $A$-module. We say $m \in M$ is *unimodular* if there exist $\phi \in M^* = \text{Hom}_A(M, A)$ such that $\phi(m) = 1$. The set of all unimodular elements of $M$ is denoted by $\text{Um}(M)$. For an ideal $J \subset A$, we denote by $E^1(A \oplus M, J)$, the subgroup of $Aut_A(A \oplus M)$ generated by all the automorphisms

$$\Delta_{a\varphi} = \begin{pmatrix} 1 & a\varphi \\ 0 & id_M \end{pmatrix} \quad \text{and} \quad \Gamma_m = \begin{pmatrix} 1 & 0 \\ m & id_M \end{pmatrix}$$

with $a \in J, \varphi \in M^*$ and $m \in M$. In particular, if $E_{r+1}(A)$ is the group generated by elementary matrices over $A$, then $E^1_{r+1}(A, J)$ denotes the subgroup of $E_{r+1}(A)$ generated by

$$\Delta_{\mathbf{a}} = \begin{pmatrix} 1 & \mathbf{a} \\ 0 & id_F \end{pmatrix} \quad \text{and} \quad \Gamma_{\mathbf{b}} = \begin{pmatrix} 1 & 0 \\ \mathbf{b}^t & id_F \end{pmatrix},$$

where $F = A^r$, $\mathbf{a} \in JF$ and $\mathbf{b} \in F$. We write $E^1(A \oplus M)$ for $E^1(A \oplus M, A)$.

By $\text{Um}^1(A \oplus M, J)$, we denote the set of all $(a, m) \in \text{Um}(A \oplus M)$ with $a \in 1 + J$, and $\text{Um}(A \oplus M, J)$ denotes the set of all $(a, m) \in \text{Um}^1(A \oplus M)$ with $m \in JM$. We write $\text{Um}_r(A, J)$ for $\text{Um}(A \oplus A^{r-1}, J)$ and $\text{Um}^1_r(A, J)$ for $\text{Um}^1(A \oplus A^{r-1}, J)$.

Let $p \in M$ and $\varphi \in M^*$ be such that $\varphi(p) = 0$. Let $\varphi_p \in End(M)$ be defined as $\varphi_p(q) = \varphi(q)p$. Then $1 + \varphi_p$ is a (unipotent) automorphism of $M$. An automorphism of $M$ of the form $1 + \varphi_p$ is called a *transvection* of $M$ if either $p \in \text{Um}(M)$ or $\varphi \in \text{Um}(M^*)$. We denote by $E(M)$, the subgroup of $Aut(M)$ generated by all transvections of $M$.

The following result is due to Bak, Basu and Rao [1, Theorem 3.10]. In [5], we proved results for $E^1(A \oplus P)$. Due to this result, we can interchange $E(A \oplus P)$ and $E^1(A \oplus P)$.

**Theorem 2.1.** *Let $A$ be a ring and $P$ a projective $A$-module of rank $\geqslant 2$. Then $E^1(A \oplus P) = E(A \oplus P)$.*

The following result follows from the definition.

**Lemma 2.2.** *Let $I \subset J$ be ideals of a ring $A$ and $P$ a projective $A$-module. Then the natural map $E^1(A \oplus P, J) \to E^1(\frac{A}{I} \oplus \frac{P}{IP}, \frac{J}{I})$ is surjective.*

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