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Vanishing ideals over complete multipartite graphs ☆

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ABSTRACT

We study the vanishing ideal of the parametrized algebraic toric set associated to the complete multipartite graph $\mathcal{G} = \mathcal{K}_{\alpha_1,\ldots,\alpha_r}$ over a finite field of order q. We give an explicit family of binomial generators for this lattice ideal, consisting of the generators of the ideal of the torus (referred to as type I generators), a set of quadratic binomials corresponding to the cycles of length 4 in $\mathcal{G}$ and which generate the *toric algebra of* $\mathcal{G}$ (type II generators) and a set of binomials of degree $q-1$ obtained combinatorially from $\mathcal{G}$ (type III generators). Using this explicit family of generators of the ideal, we show that its Castelnuovo–Mumford regularity is equal to $\max\{\alpha_1(q-2), \ldots, \alpha_r(q-2), \lceil (n-1)(q-2)/2 \rceil\}$, where $n = \alpha_1 + \cdots + \alpha_r$.

1. Introduction

The class of vanishing ideals of parameterized algebraic toric sets over a finite field was first studied by Rentería, Simis and Villarreal in [16]. Here we focus on the case when the set is parameterized by the edges of a simple graph. Let K be a finite field of order q and $\mathcal{G}$ a simple graph with n vertices $\{v_1, \ldots, v_n\}$ and nonempty edge set. Given a choice of ordering of the edges, given by a bijection $e: \{1, \ldots, s\} \to E(\mathcal{G})$, and writing $\mathbf{x}^{e(i)} = x_j x_k$ for every $\mathbf{x} = (x_1, \ldots, x_n) \in (K^*)^n$ and $e(i) = \{v_j, v_k\} \in E(\mathcal{G})$, we define the associated *algebraic toric set* as the subset of $\mathbb{P}^{s-1}$ given by:

$$X = \left\{ \left(\mathbf{x}^{e_1}, \ldots, \mathbf{x}^{e_s}\right) \in \mathbb{P}^{s-1} \colon \mathbf{x} \in \left(K^*\right)^n \right\}, \tag{1.1}$$

where we abbreviate the notation $e(i)$ to e_i.

The variety X can also be seen as the subgroup of $\mathbb{T}^{s-1} \subset \mathbb{P}^{s-1}$ given by the image of the group homomorphism $(K^*)^n \to \mathbb{T}^{s-1}$ defined by $\mathbf{x} \mapsto (\mathbf{x}^{e_1}, \ldots, \mathbf{x}^{e_s})$. The *vanishing ideal of* X, which we denote by $I(X)$, is the ideal generated by all homogeneous forms in $S = K[t_1, \ldots, t_s]$ that vanish on X. This ideal is a Cohen–Macaulay, radical, lattice ideal of codimension $s-1$ (cf. [16, Theorem 2.1]). One motivation for the study of these ideals lies in the fact that they combine the toric ideal of the edge subring of a graph, $P(\mathcal{G}) \subset K[t_1, \ldots, t_s]$, with the arithmetic of the finite field. This relation is expressed in the equality:

$$I(X) = \left(\left[P(\mathcal{G}) + \left(t_2^{q-1} - t_1^{q-1}, \ldots, t_s^{q-1} - t_1^{q-1}\right)\right] \colon (t_1 \cdots t_s)^{\infty}\right) \tag{1.2}$$

which (in particular) holds for any connected *or* bipartite $\mathcal{G}$ (cf. [16, Corollary 2.11]). Recall that $P(\mathcal{G})$ is the kernel of the epimorphism $K[t_1, \ldots, t_s] \to K[\mathcal{G}]$ given by $t_i \mapsto y^{e_i}$, where $K[\mathcal{G}]$ is the edge subring of $\mathcal{G}$, i.e., the subring of the polynomial

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ring $K[y_1,\dots,y_n]$ given by $K[G]=K[y^{e_i}\colon\ i=1,\dots,s]$. For a survey on the subject of toric ideals of edge subrings of graphs we refer the reader to [6, Chapter 5].

The properties of $I(X)$ (even in the general case of disconnected graphs) are reflected in $\mathcal{G}$ and *vice versa*. For one, the degree (or multiplicity) of $S/I(X)$ is equal to

$$\begin{cases} \left(\frac{1}{2}\right)^{\gamma-1}(q-1)^{n-m+\gamma-1}, & \text{if } \gamma\geqslant 1 \text{ and } q \text{ is odd},\\ (q-1)^{n-m+\gamma-1}, & \text{if } \gamma\geqslant 1 \text{ and } q \text{ is even},\\ (q-1)^{n-m-1}, & \text{if } \gamma=0, \end{cases} \tag{1.3}$$

where m is the number of connected components of $\mathcal{G}$, of which exactly γ are non-bipartite (cf. [15, Theorem 3.2]). Another invariant of interest is the *index of regularity* of the quotient $S/I(X)$, which, since this quotient is Cohen–Macaulay of dimension 1, coincides with the Castelnuovo–Mumford regularity. To present day knowledge, there is no single general formula expressing the regularity of $S/I(X)$ in terms of the data of $\mathcal{G}$. It is known that when $\mathcal{G}=\mathcal{C}_{2k}$, an even cycle of length $2k$, the regularity of $S/I(X)$ is $(k-1)(q-2)$ (cf. [15, Theorem 6.2]). In the case of an odd cycle, X coincides with $\mathbb{T}^{s-1}$ (cf. [16, Corollary 3.8]) — another way of seeing this, using (1.2), is that if $\mathcal{G}$ is an odd cycle then $P(\mathcal{G})=(0)$ (cf. [21]); accordingly $I(X)=(t_2^{q-1}-t_1^{q-1},\dots,t_s^{q-1}-t_1^{q-1})$ is a complete intersection (see also [8, Theorem 1]). In this case the regularity is $(s-1)(q-2)=(n-1)(q-2)$, where n (odd) is the number of vertices (and edges) of $\mathcal{G}$ (cf. [8, Lemma 1]). If $\mathcal{G}=\mathcal{K}_{a,b}$ is a complete bipartite graph, the regularity of $S/I(X)$ is given by

$$\max\{(a-1)(q-2),(b-1)(q-2)\}$$

(cf. [7, Corollary 5.4]). Recently, a formula for the regularity of $S/I(X)$ in the case of a complete graph $\mathcal{G}=\mathcal{K}_n$ was given in [9]. In this case, if $n>3$,

$$\operatorname{reg} S/I(X)=\lceil (n-1)(q-2)/2\rceil. \tag{1.4}$$

Notice that the case $\mathcal{G}=\mathcal{K}_2$ is trivial and case $\mathcal{G}=\mathcal{K}_3=\mathcal{C}_3$ was already discussed.

In this work we focus on the case of $\mathcal{G}=\mathcal{K}_{\alpha_1,\dots,\alpha_r}$, a complete multipartite graph with $\alpha_1+\cdots+\alpha_r=n$ vertices. One of our main results, Theorem 4.3, states that in this case, if $r\geqslant 3$ and $n\geqslant 4$,

$$\operatorname{reg} S/I(X)=\max\{\alpha_1(q-2),\dots,\alpha_r(q-2),\lceil (n-1)(q-2)/2\rceil\}. \tag{1.5}$$

This formula generalizes (1.4); it contains the case of the complete graph by setting $\alpha_1=\cdots=\alpha_r=1$. However, as far as the proof of Theorem 4.3 is concerned, we restrict to the case when $\mathcal{K}_{\alpha_1,\dots,\alpha_r}$ is not a complete graph. Moreover, the methods used in this work are distinctly orthogonal to those used in [9]. Our main interest being the lattice ideal $I(X)$, we rely on a precise description of a generating set of binomials to prove the statement on the regularity. In Theorem 3.3, we show that a given set of binomials generates $I(X)$. These binomials are classified into 3 classes: the binomials $t_i^{q-1}-t_j^{q-1}$, for every $i\neq j$, which, by (1.2), belong to $I(X)$ no matter which $\mathcal{G}$ we take, and are referred to as *type I generators*; the binomials $t_it_j-t_kt_l\in P(\mathcal{G})$, for each $e_ie_ke_je_l$ cycle of length 4 contained in $\mathcal{G}$, are referred to as *type II generators*; finally the *type III generators*, obtained from weighted subgraphs of $\mathcal{G}$, are described in full detail in the beginning of Section 3. Theorem 3.3 applies without restrictions on $\alpha_1,\dots,\alpha_r$. In particular, it yields a generating set for $I(X)$ in the case of a complete bipartite graph, which, despite the result on the regularity of $S/I(X)$ in [7], was missing in the literature.

This problem area has been attracting increasing interest. The field of binomial ideals has been quite explored and its general theory can be found in Eisenbud and Sturmfels article [4]. In our present setting, these binomial ideals have a remarkable application to coding theory. Associating to X an evaluation code, one can relate two of its basic parameters (the length and the dimension) to $I(X)$ by a straightforward application of the Hilbert function (cf. [11,12]); moreover a set of generators of $I(X)$ can make way to computing the Hamming distance of the code (cf. [17]). There has been substantial recent research exploring the relation to coding theory (cf. [15–18]) and also focusing on the vanishing ideal of parameterized algebraic toric sets (cf. [13,14]).

Let us describe the structure of this paper. In Section 2 we establish the definitions and notations used in the article. In that section, Lemma 2.3 provides a useful characterization of a binomial in $I(X)$ by a condition on the associated weighted subgraph of $\mathcal{G}$. In Section 3, we describe 3 families of binomials and prove that they form a generating set for $I(X)$ — Theorem 3.3. In Section 4, we prove Theorem 4.3, that states that under the assumption that $r\geqslant 3$ and $n=\alpha_1+\cdots+\alpha_r\geqslant 4$, the regularity of $S/I(X)$ is given by the integer d of formula (1.5). We show this by: (i) exhibiting a monomial in $K[t_1,\dots,t_s]$ of degree d and showing that that monomial does not belong to $I(X)+(t_1)$, where $t_1\in K[t_1,\dots,t_s]$ is a variable; (ii) and by showing that every monomial in $K[t_1,\dots,t_s]$ of degree $d+1$ is in $I(X)+(t_1)$.

For any additional information in the theory of monomial ideals and Hilbert functions, we refer to [19,20], and for graph theory we refer to [1].