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Linear systems on a class of anticanonical rational threefolds

Cindy De Volder [a], Antonio Laface [b,*,1]

[a] Department of Pure Mathematics and Computer Algebra, Krijgslaan 281, S22, B-9000 Ghent, Belgium
[b] Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

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ABSTRACT

Let X be the blow-up of the three dimensional complex projective space along r points in very general position on a smooth elliptic quartic curve $B \subset \mathbb{P}^3$ and let $L \in \mathrm{Pic}(X)$ be any line bundle. The aim of this paper is to provide an explicit algorithm for determining the dimension of $H^0(X, L)$.

1. Introduction

Let X be the blow-up of the three dimensional projective space along r points in very general position on a smooth elliptic quartic curve B. The aim of this paper is to provide an explicit algorithm for determining the dimension of $H^0(X, L)$ for any $L \in \mathrm{Pic}(X)$. This dimension depends of course on the degree and multiplicities of the general divisor of $\mathbb{P}^3$ corresponding to L. In this paper we show that in fact this number is completely determined by the values of the intersections $l_i \cdot L$ and $C \cdot L$, where the l_i's are the strict transforms of lines through pairs of the r points and C is the strict transform of B.

This work is an attempt to generalize the results of [6] to the three dimensional case by extending the techniques used in [4,3]. In [1] and [2], a higher dimensional analog of the same problem is studied under the more restrictive hypothesis that all the points lie on a rational normal curve of $\mathbb{P}^n$. It turns out that this assumption implies the finite generation of the Cox ring of the blow-up variety, while in the case analyzed by the present paper this statement is false.

The paper is organized as follows: in Section 2 we fix the necessary notation while Section 3 focuses on preliminary results regarding the intersection theory of the varieties which are needed throughout the paper. The main algorithm is explained in Section 4, where we show how starting from the linear system $\mathcal{L}$, associated to $L \in \mathrm{Pic}(X)$, one can find a fixed-component free linear system $\mathcal{L}'$ of the same dimension of $\mathcal{L}$. Then we proceed to define three different types of systems, listed in the conclusion of Section 4.2, which cover the range of all the possibilities. The dimension of a linear system in each one of these classes is given explicitly in Theorems 5.1.1, 5.2.1 and 5.3.1.

2. Notation

The aim of this section is to provide the necessary notation for linear systems defined on blow-ups of $\mathbb{P}^2$ and $\mathbb{P}^3$ and a quadric. In what follows the ground field is assumed to be algebraically closed of characteristic 0.

If D is a divisor we will denote by $\mathrm{Bs}\,|D| := \bigcap_{D' \in D} \mathrm{Supp}(D')$ the base locus of the complete linear series $|D|$.

* Corresponding author.
E-mail addresses: cindy.devolder@gmail.com (C. De Volder), alaface@udec.cl (A. Laface).
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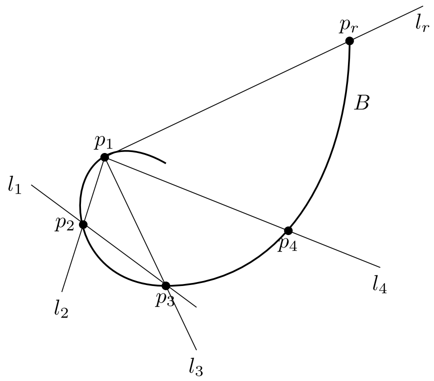

Fig. 1. The configuration of points and lines.

2.1. *The definition of X*

We start with a smooth quadric $Q \subset \mathbb{P}^3$ and a general $B \in |-K_Q|$, i.e. B is the complete intersection of two quadrics, that is an elliptic curve of degree 4. (See Fig. 1.) On this curve we choose $p_1, \dots, p_r$ points in very general position and $Z := m_1 p_1 + \cdots + m_r p_r$ is the zero dimensional subscheme of r multiple points with m_i non-negative integers and associated ideal sheaf $\mathcal{I}_Z = \mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_r}^{m_r}$. With abuse of notation we denote by

$$\mathcal{L}_{\mathbb{P}^3}(d; m_1, \dots, m_r)$$

both the sheaf $\mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z$ and its associated linear system.

The *expected dimension* of such a linear system $\mathcal{L}$ is $\mathrm{edim}(\mathcal{L}) := \max\{-1, v(\mathcal{L})\}$, where

$$v(\mathcal{L}) = \binom{d+3}{3} - \sum_{i=1}^{r} \binom{m_i+2}{3} - 1,$$

is the *virtual dimension*.

Let $\pi : X \to \mathbb{P}^3$ be the blow-up map of $\mathbb{P}^3$ along $p_1, \dots, p_r$, then we will denote by H the pull-back of a plane, by E_i the exceptional divisor corresponding to p_i and by C the strict transform of B. The elements of $\mathcal{L}_{\mathbb{P}^3}(d; m_1, \dots, m_r)$ correspond to those of the complete linear system

$$\mathcal{L}_X(d; m_1, \dots, m_r) := |dH - m_1 E_1 - \cdots - m_r E_r|$$

defined on X. As before we will use the same notation for the linear system and its associated invertible sheaf. A divisor $dH - m_1 E_1 - \cdots - m_r E_r$ on X is *standard* if

$$d \geqslant m_1 \geqslant \cdots \geqslant m_r \geqslant 0 \quad \text{and} \quad 2d \geqslant m_1 + \cdots + m_4,$$

and it is *almost standard* if it becomes standard after reordering its multiplicities. In the same way we will say that the linear systems $\mathcal{L}_X(d; m_1, \dots, m_r)$ or $\mathcal{L}_{\mathbb{P}^3}(d; m_1, \dots, m_r)$ are standard or almost standard.

2.2. *The blow-up of Q*

We have the following commutative diagram

$$\begin{array}{ccc} Q_r & \longrightarrow & X \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle \pi} \\ Q & \longrightarrow & \mathbb{P}^3 \end{array}$$

where π_Q is the blow-up map of Q at the p_i's and $e_i := E_i \cap Q_r$ is the exceptional divisor on Q_r corresponding to p_i. Let h_1, h_2 be the pull-back of the two rulings of Q. We use the notation

$$\mathcal{L}_Q(a, b; m_1, \dots, m_r) := |ah_1 + bh_2 - m_1 e_1 - \cdots - m_r e_r|$$

where $ah_1 + bh_2 = \pi_Q^* \mathcal{O}(a, b)$ to denote both the linear system and its corresponding invertible sheaf.