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Toward weakly enriched categories: co-Segal categories

Hugo V. Bacard [1]

Laboratoire J.A. Dieudonné, UNS, France

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ABSTRACT

We introduce a new type of weakly enriched categories over a given symmetric monoidal model category $\mathscr{M}$; we call them *co-Segal categories*. Their definition derives from the philosophy of classical (enriched) Segal categories. The purpose of this paper is to expose the theory and give the first results on the homotopy theory of these structures. One of the motivations of developing such theory, was to have an alternative definition of higher linear categories following Segal-like methods.

1. Introduction

In this paper we pursue the idea initiated in [3], of having a theory of weakly enriched categories over a symmetric monoidal model category $\mathscr{M} = (\underline{M}, \otimes, I)$. We introduce the notion of co-Segal $\mathscr{M}$-category. The main idea is to replace the composition law:

$$\mathcal{C}(A,B) \otimes \mathcal{C}(B,C) \longrightarrow \mathcal{C}(A,C)$$

by the following diagram.

$$\begin{array}{ccc} & & \mathcal{C}(A,C) \\ & & \downarrow \wr \\ \mathcal{C}(A,B) \otimes \mathcal{C}(B,C) & \longrightarrow & \mathcal{C}(A,B,C) \end{array}$$

In that diagram the vertical map $\mathcal{C}(A,C) \longrightarrow \mathcal{C}(A,B,C)$ is required to be a *weak equivalence* in $\mathscr{M}$.

As one can see, if this weak equivalence is an isomorphism or an identity (the strict case) then we will have a classical composition and everything is as usual. In the non-strict case, one gets a weak composition given by any choice of a weak inverse of that vertical map.

The previous diagram is obtained by 'reversing the morphisms' in the Segal situation, hence the terminology 'co-Segal'. The diagrams hereafter outline this idea.

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$$\begin{array}{ccc} & & \mathcal{C}(A,C) \\ & & \uparrow \\ \mathcal{C}(A,B)\otimes\mathcal{C}(B,C) & \xleftarrow{\sim} & \mathcal{C}(A,B,C) \end{array} \qquad\qquad \begin{array}{ccc} & & \mathcal{C}(A,C) \\ & & \downarrow \wr \\ \mathcal{C}(A,B)\otimes\mathcal{C}(B,C) & \longrightarrow & \mathcal{C}(A,B,C) \end{array}$$

In a Segal category — In a co-Segal category

If the tensor product $\otimes$ of the category $\mathscr{M} = (\underline{M}, \otimes, I)$ is different from the cartesian product $\times$ e.g $\mathscr{M}$ is a Tannakian category, the so called *Segal map* $\mathcal{C}(A, B, C) \longrightarrow \mathcal{C}(A, B) \otimes \mathcal{C}(B, C)$ appearing in the Segal situation is not 'natural'; it's a map going into a product where there is no *a priori* a way to have a projection on each factor. The co-Segal formalism was introduced precisely to bypass this problem.

In [3], following an idea introduced by Leinster [25], we define a Segal enriched category $\mathcal{C}$ having a set of objects X, as a *colax* morphism of 2-categories

$$\mathcal{C} : \mathscr{P}_{\overline{X}} \longrightarrow \mathscr{M},$$

satisfying the usual Segal conditions. As we shall see a co-Segal category is defined as a *lax* morphism of 2-categories

$$\mathcal{C} : (\mathbb{S}_{\overline{X}})^{2\text{-op}} \longrightarrow \mathscr{M},$$

satisfying the co-Segal conditions (Definition 3.4). Here $\mathscr{P}_{\overline{X}}$ is a 2-category over Δ^+ while $\mathbb{S}_{\overline{X}} \subset \mathscr{P}_{\overline{X}}$ is over Δ^+_{epi}. These 2-categories are probably examples of what we called *locally Reedy* 2*-category*, that is, a 2-category such that each category of 1-morphisms is a Reedy category and the composition is compatible with the Reedy structures.

To develop a homotopy theory of these co-Segal categories we follow the same philosophy as for Segal categories; that is, we consider the more general objects consisting of lax morphisms $\mathcal{C} : (\mathbb{S}_{\overline{X}})^{2\text{-op}} \longrightarrow \mathscr{M}$ without demanding the co-Segal conditions yet; these are called *co-Segal precategories*.

As X runs through **Set** we have a category $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$ of all *co-Segal precategories* with morphisms between them. We have a natural Grothendieck bifibration $\text{Ob} : \mathscr{M}_{\mathbb{S}}(\mathbf{Set}) \longrightarrow \mathbf{Set}$.

In this paper we present the general idea of the theory together with some results on the homotopy theory of $\mathscr{M}_{\mathbb{S}}(\mathbf{Set})$. We include only some of these results in order to avoid a too much long paper. The remaining results will appear in another paper, but few of them can already be found in [4].

Plan of the paper

We begin the paper by recalling the definition of a lax diagram in a 2-category $\mathscr{M}$, which is simply a lax functor of 2-category in the sense of Bénabou [6]. We recall in particular, that $\mathscr{M}$-categories are special cases of lax diagrams as implicitly stated by Bénabou [6] and later observed by Street [35].

In Section 3 we introduce the language of co-Segal categories starting with an overview of the one-object case. We start by exposing how the co-Segal formalism fits in the historical problem of *homotopy transfer of algebraic structure* (see Proposition 3.1).

Then we consider the notion of $\mathbb{S}$-diagram in $\mathscr{M}$ which correspond to *co-Segal precategory* (Definition 3.3). And we define a co-Segal category to be an $\mathbb{S}$-diagram satisfying the *co-Segal conditions* (Definition 3.5). After giving some definitions we show that

- A strict co-Segal $\mathscr{M}$-category is the same thing as a strict (semi) $\mathscr{M}$-category (Proposition 3.8);
- The co-Segal conditions are stable under weak equivalences (Proposition 3.11).

In Section 4 we show that the category $\mathscr{M}_{\mathbb{S}}(X)$ of co-Segal precategories with a fixed set of objects X is:

- is cocomplete if $\mathscr{M}$ is (Theorem 4.2); and
- locally presentable if $\mathscr{M}$ is (Theorem 4.1).

These two theorems are special cases of more general theorems on algebras over operad that are not included in the present paper. We provide a direct proof instead, so that the content should be accessible to a reader who is not familiar with operads.

In Section 5 we consider the notion of locally Reedy 2-category. The main motivation for considering these 2-categories, is to provide a *direct* model structure on the category $\mathscr{M}_{\mathbb{S}}(X)$ (Corollary 5.15). The techniques we've used cover also the classical case of diagram indexed by a Reedy 1-category even though we did not expose the entire treatment here.

In Section 6 we revisit the model structure on $\mathscr{M}_{\mathbb{S}}(X)$ established in Corollary 5.15 using the fact that $\mathscr{M}_{\mathbb{S}}(X)$ is the category of algebra of some monad. The key ingredient is a lemma due to Schwede and Shipley [31]. The proof is somehow redundant because it uses the fact that we already have the Reedy model structure. But there is an independent proof that is not include in this paper for a matter of length. It shall appear separately in another paper. We show precisely that if $\mathscr{M}$ is a symmetric monoidal model category, which is cofibrantly generated then we have: