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On the parallel between normality and extremal disconnectedness [☆]

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ABSTRACT

Several familiar results about normal and extremally disconnected (classical or pointfree) spaces shape the idea that the two notions are somehow dual to each other and can therefore be studied in parallel. This paper investigates the source of this 'duality' and shows that each pair of parallel results can be framed by the 'same' proof. The key tools for this purpose are relative notions of normality, extremal disconnectedness, semicontinuity and continuity (with respect to a fixed class of complemented sublocales of the given locale) that bring and extend to locale theory a variety of well-known classical variants of normality and upper and lower semicontinuities in an illuminating unified manner. This approach allows us to unify under a single localic proof all classical insertion, as well as their corresponding extension results.

Introduction

In Real Analysis, in spaces like $\mathbb{R}$ or $\mathbb{R}^2$, we are used to a great abundance of continuous maps with real values. But there are non-trivial spaces that do not admit continuous real-valued functions other than the constant ones. The abundance of real continuous functions in a space (or locale) can be assessed by the existence of functions that indeed separate all subsets (or sublocales) that can possibly be separated, and the (separation) lemma of Urysohn characterizes those spaces and locales ("*with plenty of continuous real functions*" [9]): they are precisely the normal ones. Extremally disconnected (De Morgan) spaces or locales are also very important (see [27]).

As observed by T. Kubiak in [33], several pairs of results in classical topology like those in Table 1 characterizing the concepts of normality and extremal disconnectedness show a "remarkable duality" (in the words of [33]) between the two concepts: each pair is identical in structure but prove facts about normal spaces on one side of the pair and about extremally disconnected spaces on the other. The origin of this observation goes back to [30] and it also appears in [31] and [29, p. 301] (consult [25] for more examples of results of this kind in the setting of quasi-uniform spaces). Nevertheless the known proofs of the results in each pair are quite different in nature (and the same happens with the proofs of the results in [25]), requiring even in some cases different tools and constructions.

Our recent work in the more general localic setting (see e.g. [20,15,17]) reveals a similar picture, summarized in Table 2.

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Table 1
Characterizations of normal and extremally disconnected spaces.

Space X	NORMAL	EXTREMALLY DISCONNECTED
Urysohn's separation type lemma	Every two disjoint CLOSED subsets of X are completely separated (Urysohn [47]).	Every two disjoint OPEN subsets of X are completely separated (Gillman and Jerison [12]).
Tietze's extension type theorem	Each CLOSED subset of X is C^*-embedded (Tietze [45]).	Each OPEN subset of X is C^*-embedded (Gillman and Jerison [12]).
Katětov–Tong insertion type theorem	For every UPPER semicontinuous real function f and LOWER semicontinuous real function g satisfying $f \leqslant g$, there exists a continuous real function h such that $f \leqslant h \leqslant g$ (Katětov [28], Tong [46]).	For every LOWER semicontinuous real function f and UPPER semicontinuous real function g satisfying $f \leqslant g$, there exists a continuous real function h such that $f \leqslant h \leqslant g$ (Stone [44], Lane [34]).
Hausdorff mapping invariance type theorem	The image of X under any CLOSED continuous map is NORMAL (Hausdorff [23]).	The image of X under any OPEN continuous map is EXTREMALLY DISCONNECTED.

Table 2
Characterizations of normal and extremally disconnected locales.

Locale L	NORMAL	EXTREMALLY DISCONNECTED
Urysohn's separation type lemma	Every two disjoint CLOSED sublocales of L are completely separated.	Every two disjoint OPEN sublocales of L are completely separated.
Tietze's extension type theorem	Each CLOSED sublocale of L is C^*-embedded.	Each OPEN sublocale of L is C^*-embedded.
Katětov–Tong insertion type theorem	For every UPPER semicontinuous real function f and LOWER semicontinuous real function g satisfying $f \leqslant g$, there exists a continuous real function h such that $f \leqslant h \leqslant g$.	For every LOWER semicontinuous real function f and UPPER semicontinuous real function g satisfying $f \leqslant g$, there exists a continuous real function h such that $f \leqslant h \leqslant g$.

This shapes the idea that the two notions are somehow dual to each other and therefore may be studied in parallel; hopefully, one may even find 'dual' proofs for each pair of results. It is the aim of this paper to examine this parallel. In particular, we address the following questions:

(1) What is the source of this duality?
(2) The proofs of the results in each pair are very different in nature. Can one unify them under the same result with a single proof?
(3) There is a great variety of classical insertion type results (for several variants of normality). Can one unify them under a single general result?
(4) Can one complete Table 2 with a pointfree extension of Hausdorff mapping invariance type theorems?

The main idea will be to fix a class $\mathscr{A}$ of complemented sublocales of a locale L. Depending on the parameter $\mathscr{A}$, we introduce and study dual relative notions of normality and extremal disconnectedness (respectively *$\mathscr{A}$-normality* and *$\mathscr{A}$-disconnectedness*) and notions of $\mathscr{A}$-continuous and lower and upper $\mathscr{A}$-semicontinuous real functions on L. Taking for $\mathscr{A}$ the standard closed sublocales, one obtains the usual notions. By varying the choice of $\mathscr{A}$, we reach a wide array of examples.

Since every complemented sublocale of a space is a subspace [24], in the case that the locale L is the lattice $\mathcal{O}X$ of open subsets of some space X, these notions can be completely formulated in terms of the space X, with no reference to sublocales, and provide a unification of the most relevant classical notions in the literature [8,10,28,34–37,41,43,46] (some of them are here introduced and studied for the first time in the pointfree setting).

Our results characterize $\mathscr{A}$-normal locales and generalize all characterizations in Table 2. They hold for any class $\mathscr{A}$ that induces a Katětov relation on the lattice of all sublocales. Then the dual results for $\mathscr{A}$-disconnectedness correspond to the results for the class $\mathscr{A}^c$ of complements of elements of $\mathscr{A}$ and are therefore obtained with no extra cost. Again, this approach allows to extend and unify the most relevant classical insertion results [8,28,46,35,44].

By relativizing the notion of an extension of a real function on a sublocale to the whole locale, we obtain a relative form of Tietze's extension theorem and the corresponding dual result. In addition we also prove a relative version for the preservation of normality under localic maps that extends the Hausdorff mapping invariance type theorems of Table 1 to the pointfree setting, thus completing Table 2.

There is one important aspect of insertion and normality which is not considered in this paper, namely strict insertion [16,21] and its connection with variants of perfect normality. This will be treated in a separate paper [18].

1. Background and notation

We take the localic approach to topology. If X is a topological space, the partially ordered set $\mathcal{O}X$ of open subsets of X is a complete lattice, in which the infinite distributive law