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Gorenstein derived equivalences and their invariants ☆

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ABSTRACT

The main objective of this paper is to study the relative derived categories from various points of view. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ be a contravariantly finite subcategory of $\mathcal{A}$. One can define $\mathcal{C}$-relative derived category of $\mathcal{A}$, denoted by $\mathbb{D}^*_{\mathcal{C}}(\mathcal{A})$. The interesting case for us is when $\mathcal{A}$ has enough projective objects and $\mathcal{C} = \mathcal{GP}\text{-}\mathcal{A}$ is the class of Gorenstein projective objects, where $\mathbb{D}^*_{\mathcal{C}}(\mathcal{A})$ is known as the Gorenstein derived category of $\mathcal{A}$. We explicitly study the relative derived categories, specially over artin algebras, present a relative version of Rickard's theorem, specially for Gorenstein derived categories, provide some invariants under Gorenstein derived equivalences and finally study the relationships between relative and (absolute) derived categories.

1. Introduction

Let A and B be two rings. A and B are called derived equivalent if there is a triangulated equivalence $\mathbb{D}^b(\text{Mod-}A) \cong \mathbb{D}^b(\text{Mod-}B)$, or equivalently, if there is a triangulated equivalence $\mathbb{D}^b(\text{mod-}A) \cong \mathbb{D}^b(\text{mod-}B)$.

Let Λ be a ring. A finitely generated Λ-module T is called a (generalized) tilting module if T admits a finite length resolution by finitely generated projective modules, $\text{Ext}^i_{\Lambda}(T, T) = 0$, for all $i > 0$ and there exists an exact sequence

$$0 \to \Lambda \to T_0 \to \cdots \to T_m \to 0$$

such that $T_i \in \text{add-}T$. Let Λ be a finite dimensional artin algebra. Happel's theorem proves that Λ and Γ are derived equivalent, provided Γ is the endomorphism ring of a tilting Λ-module.

This result was soon extended to the general setting of arbitrary rings in [13]. But the converse of Happel's theorem was still open, that is, whether any triangulated equivalence between bounded derived categories, arises from a tilting module. Three years later, Rickard answered this question and developed a Morita theory for derived categories of module categories. He showed that in his theory the role of tilting modules should be replaced by the so-called tilting complexes.

Let $\mathcal{A}$ be an abelian category. The concept of the Gorenstein derived category of $\mathcal{A}$, $\mathbb{D}_{\mathcal{GP}}(\mathcal{A})$ is introduced and studied by Gao and Zhang in [17] as the Verdier quotient of the homotopy category $\mathbb{K}^*(\mathcal{A})$ with respect to the thick triangulated subcategory $\mathbb{K}^*_{\mathcal{GP}\text{-ac}}(\mathcal{A})$ of $\mathcal{GP}$-acyclic complexes. The authors of [17] proved some of the basic properties of $\mathbb{D}^*_{\mathcal{GP}}(\mathcal{A})$ and also discussed a version of Rickard's theorem for Gorenstein derived categories under certain conditions on the algebras involved.

Let $\mathcal{C}$ be a contravariantly finite subcategory of an abelian category $\mathcal{A}$. One can define the $\mathcal{C}$-relative derived category of $\mathcal{A}$, denoted by $\mathbb{D}^*_{\mathcal{C}}(\mathcal{A})$, similarly. Rickard provided a Morita theory for derived categories. We introduce and study relative

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Morita theory for Gorenstein derived categories. To this end, we introduce the notion of relative tilting complexes and show that they play a similar role as tilting complexes of Rickard. Then we focus on the invariants under Gorenstein derived equivalences. In particular, we show that the finiteness of finitistic dimension is invariant under Gorenstein derived equivalences. The relationship between derived and Gorenstein derived equivalences will be studied in certain cases.

The paper is structured as follows. After some preliminaries, Section 3 is devoted to the notion of relative derived categories. This subject has already been studied in a paper of Chen [12]. One interesting case is when $\mathcal{C} = \text{add-}M$, where M is a finitely generated module over an artin algebra Λ. In this case we denote $\mathbb{D}^b_{\text{add-}M}(\text{mod-}\Lambda)$ by $\mathbb{D}^b_M(\text{mod-}\Lambda)$. We show that

$$\mathbb{D}^b_M(\text{mod-}\Lambda) \cong \mathbb{D}^b\big(\text{mod-End}_\Lambda(M)\big).$$

This observation has various interesting consequences. Based on that, we generalize a result of Gao and Zhang, Corollary 3.7; get an easy proof of a result of Rouquier, Corollary 3.8; and also reprove a theorem of Beligiannis, Corollary 3.9. Furthermore, we get an equivalence $\mathbb{D}_M(\text{Mod-}\Lambda) \cong \mathbb{D}(\text{Mod-End}_\Lambda(M))$ for unbounded derived categories.

In Section 4, we introduce $\mathcal{C}$-relative tilting complexes and prove a relative version of Rickard's theorem. The interesting case for us is when Λ is an artin algebra and $\mathcal{C} = \mathcal{G}p\text{-}\Lambda$ is the class of Gorenstein projective Λ-modules. We introduce the notion of Gorenstein tilting modules and show that the deleted Gorenstein projective resolution of any Gorenstein tilting module is a Gorenstein tilting complex. This theory has some interesting corollaries. For example, we provide a short proof for the main theorem of [20].

Then we turn our attention to invariants under Gorenstein derived equivalences. This will be done in Section 5. Let Λ and Λ' be artin algebras that are Gorenstein derived equivalent. We show that:

- If Λ and Λ' are of finite CM-type, then
  - Λ is Gorenstein if and only if Λ' is so.
  - fin.dimΛ is finite if and only if fin.dimΛ' is so.
- If Λ and Λ' are Gorenstein, then Λ is of finite CM-type if and only if Λ' is so.

In the last part of the paper, some connections between derived and Gorenstein derived equivalences are given. For example, we show that if Λ and Λ' are derived equivalent artin algebras, then they are Gorenstein derived equivalent, provided Λ and Λ' are either Gorenstein or virtually Gorenstein of finite Cohen–Macaulay type. In our final remark, we show that the main theorem of [28] follows easily from our results, using relative derived categories.

2. Preliminaries

In general, we assume that Λ is an arbitrary ring, associative with the identity element. We let Mod-Λ, resp. mod-Λ, denote the category of all, resp. all finitely presented, right Λ-modules. Let $\mathcal{M}$ be a class of Λ-modules. We denote by Add-$\mathcal{M}$ the class of all modules that are isomorphic to a direct summand of a direct sum of modules in $\mathcal{M}$. Moreover, add-$\mathcal{M}$ denotes the class of all direct summands of finite direct sums of objects in $\mathcal{M}$. If $\mathcal{M} = \{M\}$ contains a single object, we write Add-M, resp. add-M, instead of Add-$\{M\}$, resp. add-$\{M\}$.

Let $\mathcal{X}$ be an additive category. We denote by $\mathbb{C}(\mathcal{X})$ the category of complexes in $\mathcal{X}$; the objects are complexes and morphisms are genuine chain maps. We write the complexes cohomologically, so an object of $\mathbb{C}(\mathcal{X})$ is of the following form

$$\cdots \to X^{n-1} \xrightarrow{\partial^{n-1}} X^n \xrightarrow{\partial^n} X^{n+1} \to \cdots.$$

If $\mathcal{A} = \text{Mod-}\Lambda$ is the category of (right) Λ-modules, we write $\mathbb{C}(\Lambda)$ for $\mathbb{C}(\text{Mod-}\Lambda)$. It is known that in case $\mathcal{A}$ is additive (resp. abelian) then so is $\mathbb{C}(\mathcal{A})$. In particular, $\mathbb{C}(\Lambda)$ is an abelian category.

The homotopy category of $\mathcal{X}$ will be denoted by $\mathbb{K}(\mathcal{X})$. Its objects are the same as $\mathbb{C}(\mathcal{X})$ and morphisms are the homotopy classes of morphisms of complexes.

2.1. Let $\mathcal{X}$ be an additive category. A complex $\mathbf{X} \in \mathbb{C}(\mathcal{X})$ is called acyclic if $\text{H}^n(\mathbf{X}) = 0$, for all $n \in \mathbb{Z}$. $\mathbf{X}$ is called $\mathcal{X}$-totally acyclic if the induced complexes $\mathcal{X}(\mathbf{X}, Y)$ and $\mathcal{X}(Y, \mathbf{X})$ of abelian groups are acyclic, for all $Y \in \mathcal{X}$. In this case, the syzygies of $\mathbf{X}$ are called $\mathcal{X}$-Gorenstein projective objects. We let $\mathcal{G}(\mathcal{X})$ denote the class of all $\mathcal{X}$-Gorenstein projective objects.

Let $\mathbb{C}_{\text{ac}}(\mathcal{X})$, resp. $\mathbb{C}_{\text{tac}}(\mathcal{X})$, denote the full subcategory of $\mathbb{C}(\mathcal{X})$ consisting of acyclic, resp. totally acyclic, complexes. Moreover, the triangulated subcategory of $\mathbb{K}(\mathcal{X})$ consisting of acyclic, resp. totally acyclic, complexes will be denoted by $\mathbb{K}_{\text{ac}}(\mathcal{X})$, resp. $\mathbb{K}_{\text{tac}}(\mathcal{X})$.

Let $\mathcal{A}$ be an abelian category. If $\mathcal{X} = \text{Prj-}\mathcal{A}$, resp. $\mathcal{X} = \text{Inj-}\mathcal{A}$, is the class of projectives, resp. injectives, in $\mathcal{A}$, the objects of $\mathbb{K}_{\text{tac}}(\mathcal{X})$ will be called totally acyclic complexes of projectives, resp. totally acyclic complexes of injectives. The full subcategory of $\mathbb{K}(\mathcal{A})$ consisting of totally acyclic complexes of projectives, resp. injectives, will be denoted by $\mathbb{K}_{\text{tac}}(\text{Prj-}\mathcal{A})$, resp. $\mathbb{K}_{\text{tac}}(\text{Inj-}\mathcal{A})$. The objects of $\mathcal{G}(\text{Prj-}\mathcal{A})$, resp. $\mathcal{G}(\text{Inj-}\mathcal{A})$, are called Gorenstein projectives, resp. Gorenstein injectives.

In these special cases we denote $\mathcal{G}(\text{Prj-}\mathcal{A})$, resp. $\mathcal{G}(\text{Inj-}\mathcal{A})$, by $\mathcal{GP}\text{-}\mathcal{A}$, resp. $\mathcal{GI}\text{-}\mathcal{A}$. In case $\mathcal{A} = \text{Mod-}\Lambda$, we abbreviate the notions to $\mathcal{GP}\text{-}\Lambda$ and $\mathcal{GI}\text{-}\Lambda$. We set $\mathcal{G}p\text{-}\Lambda = \mathcal{GP}\text{-}\Lambda \cap \text{mod-}\Lambda$ and $\mathcal{G}i\text{-}\Lambda = \mathcal{GI}\text{-}\Lambda \cap \text{mod-}\Lambda$.

It is known [22,33] that $\mathcal{G}(\mathcal{G}(\text{Prj-}\Lambda)) = \mathcal{G}(\text{Prj-}\Lambda)$ and $\mathcal{G}(\mathcal{G}(\text{prj-}\Lambda)) = \mathcal{G}(\text{prj-}\Lambda)$, where prj-$\Lambda$ is the class of all finitely generated projective Λ-modules. We shall use these facts later.