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Local cohomology of bigraded Rees algebras and normal Hilbert coefficients

Shreedevi K. Masuti [1], J.K. Verma *

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, 400076, India

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Let $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring of dimension 2 with infinite residue field and $\bar{I}$ be the integral closure of an ideal I in R. Necessary and sufficient conditions are given for $\overline{I^{r+1}J^{s+1}} = a\overline{I^rJ^{s+1}} + b\overline{I^{r+1}J^s}$ to hold for all $r \geqslant r_0$ and $s \geqslant s_0$ in terms of vanishing of $[H^2_{(at_1,bt_2)}(\overline{\mathcal{R}'}(I,J))]_{(r_0,s_0)}$, where $a \in I, b \in J$ is a good joint reduction of the filtration $\{\overline{I^rJ^s}\}$. This is used to derive a theorem due to Rees on normal joint reduction number zero. The vanishing of $\bar{e}_2(IJ)$ is shown to be equivalent to Cohen–Macaulayness of $\overline{\mathcal{R}}(I,J)$.

1. Introduction

Let R be a commutative ring. Let I be an ideal of R. An element $x \in R$ is called integral over I if x satisfies the equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

for some $a_i \in I^i$, $i = 1, 2, \ldots, n$. The set $\bar{I}$ of elements that are integral over I is an ideal, called the *integral closure* of I. If $I = \bar{I}$ then I is called *complete* or *integrally closed*. A Noetherian local ring $(R, \mathfrak{m})$ is said to be *analytically unramified* if its $\mathfrak{m}$-adic completion is reduced. Let I be an $\mathfrak{m}$-primary ideal. In an analytically unramified local ring R of dimension d, there exists a polynomial $\overline{P}_I(x) \in \mathbb{Q}[x]$ of degree d called the *normal Hilbert polynomial* of I, such that

$$\lambda\big(R/\overline{I^n}\big) = \overline{P}_I(n) \quad \text{for } n \gg 0,$$

where $\lambda(M)$ denotes the length of an R-module M [12, Theorem 1.4] and [13, Theorem 1.1]. We write

$$\overline{P}_I(n+1) = \bar{e}_0(I)\binom{n+d}{d} - \bar{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\bar{e}_d(I)$$

for some integers $\bar{e}_i(I)$, $i = 0, \ldots, d$. The coefficient $\bar{e}_0(I) = e(I)$ is called the *multiplicity of* I. P.B. Bhattacharya [1, Theorem 8] showed that for $\mathfrak{m}$-primary ideals I and J in a Noetherian local ring $(R, \mathfrak{m})$ of dimension d there exist integers $e_{(i,j)}(I, J)$ such that for large r, s

$$\lambda\big(R/I^rJ^s\big) = \sum_{i+j \leqslant d} (-1)^{d-(i+j)} e_{(i,j)}(I,J)\binom{r+i-1}{i}\binom{s+j-1}{j}.$$

* Corresponding author.
E-mail addresses: shreedevi@math.iitb.ac.in (S.K. Masuti), jkv@math.iitb.ac.in (J.K. Verma).

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Let $\mathbb{N}$ denote the set of nonnegative integers. Rees studied the numerical function

$$\overline{H}_{I,J} : \mathbb{N}^2 \longrightarrow \mathbb{N} \quad \text{defined as} \quad \overline{H}_{I,J}(r,s) = \lambda\big(R/\overline{I^r J^s}\big).$$

He proved [15] that there exists a polynomial $\overline{P}_{I,J}(x,y) \in \mathbb{Q}[x,y]$ of total degree d such that $\overline{P}_{I,J}(r,s) = \overline{H}_{I,J}(r,s)$ for $r,s \gg 0$ in an analytically unramified local ring. We write

$$\overline{P}_{I,J}(x,y) = \sum_{i+j \leqslant d} (-1)^{d-(i+j)} \bar{e}_{(i,j)}(I,J) \binom{x+i-1}{i} \binom{y+j-1}{j}$$

for some integers $\bar{e}_{(i,j)}(I,J)$. Let

$$\text{extended Rees ring of } \mathcal{I} = \{I^r J^s\} \quad \text{be } \mathcal{R}'(I,J) = \bigoplus_{r,s \in \mathbb{Z}} I^r J^s t_1^r t_2^s,$$

$$\text{extended Rees ring of } \mathcal{I} = \{\overline{I^r J^s}\} \quad \text{be } \overline{\mathcal{R}'}(I,J) = \bigoplus_{r,s \in \mathbb{Z}} \overline{I^r J^s} t_1^r t_2^s.$$

One of the main objectives of this paper is to understand an interesting theorem of Rees [15] which asserts that $\bar{e}_2(IJ) = \bar{e}_2(I) + \bar{e}_2(J)$ for $\mathfrak{m}$-primary ideals I and J in an analytically unramified Cohen–Macaulay local ring of dimension 2 with infinite residue field if and only if for all $r,s \geqslant 0$,

$$\overline{I^{r+1} J^{s+1}} = a\overline{I^r J^{s+1}} + b\overline{I^{r+1} J^s}, \tag{1.1}$$

where $a \in I$, $b \in J$ is a good joint reduction of I and J. See Section 2. As a consequence Rees proved that product of complete ideals is complete in 2-dimensional pseudo-rational local rings. Rees showed that regular local rings are pseudo-rational and thus generalized Zariski's product theorem. Another consequence of Rees's theorem is a formula for the Hilbert polynomial of an integrally closed ideal in a two-dimensional regular local ring. We generalize Rees's theorem:

Theorem 1.2. *Let R be an analytically unramified Cohen–Macaulay local ring of dimension 2 with infinite residue field and (a,b) be a good joint reduction of the filtration $\{\overline{I^r J^s}\}$. Let $r_0, s_0 \geqslant 0$. Then following statements are equivalent:*

(1) $\bar{e}_2(I) + \bar{e}_2(J) - \bar{e}_2(IJ) = \lambda(R/\overline{I^{r_0} J^{s_0}}) - g_{r_0}(I,J) - h_{s_0}(I,J) - r_0 s_0 e(I|J)$,
(2) $[H^2_{(at_1,bt_2)}(\overline{\mathcal{R}'}(I,J))]_{(r_0,s_0)} = 0$,
(3) $\overline{I^{r+1} J^{s+1}} = a\overline{I^r J^{s+1}} + b\overline{I^{r+1} J^s}$ *for* $r \geqslant r_0$, $s \geqslant s_0$,

where $e(I|J) = e_{(1,1)}(I,J)$ and $g_{r_0}(I,J), h_{s_0}(I,J)$ satisfy

$$\lambda\big(\overline{J^s}/\overline{I^{r_0} J^s}\big) = e(I|J) r_0 s + g_{r_0}(I,J) \quad \textit{for } s \gg 0 \quad \textit{and}$$

$$\lambda\big(\overline{I^r}/\overline{I^r J^{s_0}}\big) = e(I|J) r s_0 + h_{s_0}(I,J) \quad \textit{for } r \gg 0.$$

We derive a formula for $\lambda([H^2_{(at_1,bt_2)}(\overline{\mathcal{R}'}(I,J))]_{(r,s)})$ in terms of Hilbert coefficients. The above theorem also gives a cohomological interpretation of Rees's theorem since

$$\lambda\big(\big[H^2_{(at_1,bt_2)}\big(\overline{\mathcal{R}'}(I,J)\big)\big]_{(0,0)}\big) = \bar{e}_2(I) + \bar{e}_2(J) - \bar{e}_2(IJ).$$

We will gather some preliminary results about existence of good joint reductions in Section 2.

In Section 3, we calculate the local cohomology of bigraded extended Rees algebra of the filtration $\{\overline{I^r J^s}\}$.

In Section 4, a new proof of Rees's theorem and its generalization are obtained. We give an application of Theorem 1.2 to the normal reduction number of an ideal by deriving a result of T. Marley [11, Corollary 3.8] which asserts that $\bar{r}(I) \leqslant k+1$ if and only if $\lambda(R/\overline{I^k}) = \overline{P}_I(k)$.

In Section 5, we study vanishing of $\bar{e}_2(IJ)$. We prove that the vanishing of $\bar{e}_2(IJ)$ is equivalent to Cohen–Macaulayness of $\overline{\mathcal{R}}(I,J)$, where

$$\overline{\mathcal{R}}(I,J) = \bigoplus_{r,s \geqslant 0} \overline{I^r J^s} t_1^r t_2^s = \text{the Rees ring of the filtration } \mathcal{I} = \{\overline{I^r J^s}\}.$$

We refer [3] for all undefined terms.