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Twisted dendriform algebras and the pre-Lie Magnus expansion

Kurusch Ebrahimi-Fard [a,*], Dominique Manchon [b]

[a] Department de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, E-50009 Zaragoza, Spain

[b] Université Blaise Pascal, C.N.R.S.-UMR 6620 63177 Aubière, France

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ABSTRACT

In this paper an application of the recently introduced pre-Lie Magnus expansion to Jackson's q-integral and q-exponentials is presented. Twisted dendriform algebras, which are the natural algebraic framework for Jackson's q-analogs, are introduced for that purpose. It is shown how the pre-Lie Magnus expansion is used to solve linear q-differential equations. We also briefly outline the theory of linear equations in twisted dendriform algebras.

1. Introduction

This article is a continuation of our recent work on the pre-Lie Magnus expansion and linear dendriform equations. In [17] we use Loday's dendriform algebra [29] to describe the natural pre-Lie algebra structure underlying the classical Magnus expansion [30]. This point of view motivated us to explore the solution theory of a particular class of linear dendriform equations [18] appearing in the contexts of different applications, such as for instance perturbative renormalization in quantum field theory. Our results fit into recent developments exploring algebro-combinatorial aspects related to Magnus' work [9,10,19,24]. In both Refs. [17,18] we mentioned Jackson's q-integral and linear q-difference equations as a particular setting in which our results apply.

In the current paper we would like to explore in more detail linear q-difference equations and their (q-)exponential solutions in terms of a possible q-analog of the pre-Lie Magnus expansion. We are led to introduce the notion of twisted dendriform algebra as a natural setting for this work.

The paper is organized as follows. Section 2 recalls several mathematical structures needed in what follows such as pre-Lie algebras and Rota–Baxter algebras. Bits of q-calculus, linear q-difference equations as well as Jackson's q-analog of the Riemann integral are also introduced. Section 3 contains the definition of unital twisted dendriform algebra and extends results from earlier work [17,18] to this setting. Finally in Section 4 we explain how the pre-Lie Magnus expansion in this setting gives rise to a version of the Magnus expansion involving Jackson integrals. In an Appendix we briefly remark on a link of our results to finite difference operators.

2. Preliminaries

In this paragraph we summarize some well-known facts on various algebraic structures that will be of use in the rest of the paper, namely pre-Lie algebras, Rota–Baxter algebras and the pre-Lie Magnus expansion. We also give a brief account of Jackson integrals, q-difference operators and linear q-difference equations.

* Corresponding author.
E-mail addresses: kef@unizar.es, kurusch.ebrahimi-fard@uha.fr (K. Ebrahimi-Fard), manchon@math.univ-bpclermont.fr (D. Manchon).
URLs: http://www.th.physik.uni-bonn.de/th/People/fard/ (K. Ebrahimi-Fard), http://math.univ-bpclermont.fr/~manchon/ (D. Manchon).

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2.1. Pre-Lie algebras

Let us start by recalling the notion of pre-Lie algebra [1,7,8]. A *left (right) pre-Lie algebra* is a k-vector space P with a bilinear product $\rhd$ ($\lhd$) such that for any $a, b, c \in P$:

$$(a \rhd b) \rhd c - a \rhd (b \rhd c) = (b \rhd a) \rhd c - b \rhd (a \rhd c), \tag{1}$$

$$(a \lhd b) \lhd c - a \lhd (b \lhd c) = (a \lhd c) \lhd b - a \lhd (c \lhd b). \tag{2}$$

Observe that for any left pre-Lie product $\rhd$ the product $\lhd$ defined by $a \rhd b := -b \lhd a$ is right pre-Lie. One shows easily that the left pre-Lie identity rewrites as

$$L_\rhd([a, b]) = [L_\rhd(a), L_\rhd(b)],$$

where the left multiplication map $L_\rhd(a) : P \to P$ is defined by $L_\rhd(a)(b) := a \rhd b$, and where the bracket on the left-hand side is defined by $[a, b] := a \rhd b - b \rhd a$. As a consequence this bracket satisfies the Jacobi identity and hence defines a Lie algebra on P, denoted by $\mathcal{L}_P$: in order to show that (see [1]) we can add a unit $\mathbf{1}$ to the left pre-Lie algebra by considering the vector space $\overline{P} = P \oplus k.\mathbf{1}$ together with the extended product $(a + \alpha\mathbf{1}) \rhd (b + \beta\mathbf{1}) = a \rhd b + \alpha b + \beta a + \alpha\beta\mathbf{1})$, which is still left pre-Lie. The map $a \mapsto L_\rhd(a)$ is then obviously an injective map from P into $\mathrm{End}\overline{P}$, which preserves both brackets.

An easy but important observation is that a commutative left (or right) pre-Lie algebra is necessarily associative; see [1].

2.2. Rota–Baxter algebras

Recall [4,2,36] that a *Rota–Baxter algebra* (over a field k) *of weight* $\lambda \in k$ is an associative k-algebra A endowed with a k-linear map $R : A \to A$ subject to the following relation:

$$R(a)R(b) = R\big(R(a)b + aR(b) + \lambda ab\big). \tag{3}$$

The map R is called a *Rota–Baxter operator of weight*[1] λ. The map $\widetilde{R} := -\lambda id - R$ also is a weight λ Rota–Baxter map on A. Both, the image of R and $\tilde{R}$ form subalgebras in A. Recalling the classical integration by parts rule one realizes that the ordinary Riemann integral, $If(x) := \int_0^x f(y)dy$, is a weight zero Rota–Baxter map. Let (A, R) be a Rota–Baxter algebra of weight λ. Define

$$a *_R b := R(a)b + aR(b) + \lambda ab. \tag{4}$$

The vector space underlying A, equipped with the product $*_R$ is again a Rota–Baxter algebra of weight λ and $R(a *_R b) = R(a)R(b)$. Now observe that Rota–Baxter algebras of any other kind than associative still make sense, with the same definition except that the associative product is replaced by a more general bilinear product. Indeed, let A be an associative Rota–Baxter algebra of weight λ and define

$$a \rhd_R b := [R(a), b] - \lambda ba = R(a)b - bR(a) - \lambda ba. \tag{5}$$

The vector space underlying A, equipped with the product $\rhd_R$ is a Rota–Baxter left pre-Lie algebra in that sense. Observe that

$$R(a *_R b) + R(b \rhd_R a) = R(R(a)b) + R(R(b)a).$$

2.3. The pre-Lie Magnus expansion

Let A be the algebra of piecewise smooth functions on $\mathbb{R}$ with values in some associative algebra, e.g. square matrices. Recall Magnus' expansion [30], which allows us to write the formal solution of the initial value problem:

$$\dot{X}(t) = U(t)X(t), \qquad X(0) = \mathbf{1}, \quad U \in \lambda A[[\lambda]] \tag{6}$$

as $X(t) = \exp(\Omega(U)(t))$. The function $\Omega(U)(t)$ solves the differential equation:

$$\dot{\Omega}(U)(t) = \frac{ad_\Omega}{e^{ad_\Omega} - 1}(U)(t) = U(t) + \sum_{n>0} \frac{B_n}{n!} ad_\Omega^n U(t), \tag{7}$$

with initial value $\Omega(U)(0) = 0$, where the B_n are the Bernoulli numbers. Here, e^{ad_Ω} denotes the usual formal exponential operator series. We refer the reader to the recent works [24,25,5] for more details on Magnus' result and its wide spectrum

[1] The early papers on Rota–Baxter algebras use the opposite sign convention for the weight. We choose the convention initiated in [20,21]. In particular, idempotent Rota–Baxter operators are of weight −1.