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Conjugation in Brauer algebras and applications to character theory

Armin Shalile

Institute for Algebra and Number Theory, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

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ABSTRACT

The Brauer algebra has a basis of diagrams and these generate a monoid H consisting of scalar multiples of diagrams. Following a recent paper by Kudryavtseva and Mazorchuk, we define and completely determine three types of conjugation in H. We are thus able to define Brauer characters for Brauer algebras which share many of the properties of Brauer characters defined for finite groups over a field of prime characteristic. Furthermore, we reformulate and extend the theory of characters for Brauer algebras as introduced by Ram to the case when the Brauer algebra is not semisimple.

0. Introduction

In [21], Issai Schur proved a fundamental link between representations of symmetric groups and general linear groups known as Schur–Weyl duality. This duality arises from the following double centralizer property: Let r and n be natural numbers, denote by Σ_r the symmetric group on r letters, by $GL_n(\mathbb{C})$ the general linear group of invertible $n \times n$ matrices over $\mathbb{C}$ and suppose E is an n-dimensional complex vector space. Then Σ_r and $GL_n(\mathbb{C})$ act on the r-fold tensor space $E^{\otimes r}$ by place permutations and diagonal extension of the natural action on E, respectively. The $\mathbb{C}$-endomorphisms induced by the action of Σ_r and $GL_n(\mathbb{C})$ on $E^{\otimes r}$ are precisely those $\mathbb{C}$-endomorphism which commute with $GL_n(\mathbb{C})$ and Σ_r, respectively.

In [1], Richard Brauer defined an algebra which replaces the symmetric group in Schur–Weyl duality if one replaces the general linear group by its orthogonal and symplectic subgroups. This algebra is known today as the Brauer algebra and will be defined in Section 1.1.

Even though Richard Brauer already defined these algebras in 1937, very little is known about them — despite extensive efforts. The number of simple modules over arbitrary fields was only determined in 1996 by Graham and Lehrer, see [5]. Furthermore, it was not before 2005 that the question of when Brauer algebras are semisimple was solved in full generality, see [20]. Recently, however, there has been some significant progress. In [3], the blocks of the Brauer algebra have been determined over a field of characteristic 0 and in [16] the decomposition numbers over the complex field have been determined. Furthermore, a different way to compute the decomposition matrices in non-describing characteristic was given in [23].

The representation theory of Brauer algebras is in many respects similar to the representation theory of the symmetric group. Brauer algebras contain the group algebra of the symmetric group both as a subalgebra and as a quotient. Furthermore, the Brauer algebra is an iterated inflation of symmetric group algebras in the sense of [11]. The similarity to symmetric group algebras has been frequently exploited in the study of Brauer algebras. In [18], Arun Ram developed a theory of ordinary characters. Furthermore, Hartmann and Paget have defined analogues of permutation modules and Young modules, see [6]. We will continue this agenda by determining analogues of conjugacy classes and Brauer characters for Brauer algebras which is the main result of this paper.

Summary of this paper: Let F be a field and suppose $\delta \in F$ is a fixed parameter. The Brauer algebra has a basis of diagrams and the multiplication of two diagrams yields another diagram, potentially scaled by a power of the parameter

E-mail address: shalile@mathematik.uni-stuttgart.de.

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δ. Since multiplication is associative, this gives rise to a possibly infinite semigroup H generated by diagrams and their products. Following [13], we define and completely describe three notions of conjugacy in H extending results from [12].

The first type is via conjugation by the group of units in H. The group of units of H can be identified with permutations and potentially scalar multiples of permutations in case δ has finite order. This is the strongest of the three notions of conjugacy and will imply the other types of conjugacy. In order to determine the conjugacy classes, we introduce a new invariant, the generalized cycle type, and show that two diagrams are conjugate by a permutation if and only if they have equivalent generalized cycle types, see Theorem 2.6.

The second type of conjugation, called (uv, vu)-conjugation, arises from taking the transitive closure of the following relation on H: Two elements $h_1, h_2 \in H$ are related if there are $u, v \in H$ such that $h_1 = uv$ and $h_2 = vu$. It turns out that the conjugacy classes in this case have a nice description in terms of the cycle type, an invariant introduced by Arun Ram in [18]. However, proving this poses far more challenges than the first kind of conjugation. We will first show that diagrams with the same cycle type are (uv, vu)-conjugate by using the control the generalized cycle type gives us over the combinatorics.

As a consequence of the study of the combinatorics, we get some information about eigenvalues of matrix representations. This will eventually allow us to define Brauer characters for Brauer algebras which are a generalization of the Brauer character of a finite group over fields of prime characteristic, see Definition 5.2. Furthermore, we will reformulate and extend the character theory studied by Ram in the semisimple case to the non-semisimple case. In Theorem 5.10 and Theorem 6.6, we will show that the relationship between characters of Brauer algebras and symmetric groups, observed by Ram in [18], also carries over to the non-semisimple case and even to Brauer characters. This will imply that in order to compute characters, and thus decomposition matrices, it is sufficient to understand the restriction of the simple Brauer algebra modules to the symmetric group.

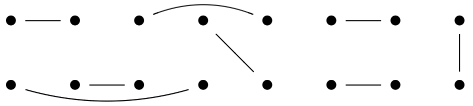
In Theorem 6.11 we use our knowledge of characters to completely describe the third kind of conjugation: Two elements h_1 and h_2 are χ-conjugate if and only if for all Brauer algebra characters χ we have $\chi(h_1) = \chi(h_2)$. Since χ-conjugacy is the weakest of the three types of conjugacy, this will also allow us to complete our study of (uv, vu)-conjugation, see Corollary 6.12.

We remark that, similar to the case of finite groups, the knowledge of conjugacy classes gives a lot of information about the center of the Brauer algebra. In particular, we are able to provide an explicit algorithm for the determination of a basis of the center for all but finitely many values of the parameter, see [22].

1. Preliminaries

1.1. Definition of Brauer algebras

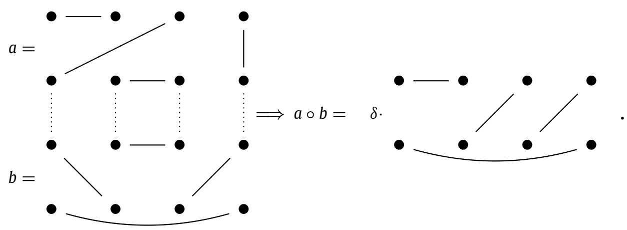
Let r be a natural number. By a diagram on $2r$ dots we mean a $2 \times r$-array of dots, that is two rows of r dots each, and each dot is connected by an edge to exactly one other dot distinct from itself. An example of a diagram on 2×8 dots is



We denote by D_r the set of all diagrams on $2r$ dots and usually omit the index if r is clear from the context. A horizontal arc is an edge connecting two points in the same row, while a through arc is an edge connecting two dots in different rows. A vertical arc is a special case of a through arc connecting two dots in the same column.

Given a field F and a parameter $\delta \in F$, it is possible to compose diagrams on the same number of dots by a process called concatenation, see for example Section 2 of [24]. To concatenate two diagrams a and b, write a on top of b and connect adjacent rows. The concatenation $a \circ b$ is the diagram obtained by deleting the loops of this construction, premultiplied by as many powers of δ as there were loops removed.

For example



Definition 1.1. Let F be a field, fix some $\delta \in F$ and let r be a natural number (we adopt the convention that $0 \notin \mathbb{N}$ throughout this paper). Define the Brauer algebra $B_r(\delta)$ to be the associative F-algebra with basis D_r and multiplication given by concatenation.

Remark 1.2. (i) If F is a field of rational polynomials in a single indeterminate x and $\delta = x$, then we get the so called generic Brauer algebra.