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Support varieties for Frobenius kernels of classical groups

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ABSTRACT

Let G be a classical simple algebraic group over an algebraically closed field k of characteristic $p > 0$, and denote by $G_{(r)}$ the r-th Frobenius kernel of G. We show that for p large enough, the support variety of a simple G-module over $G_{(r)}$ can be described in terms of support varieties of simple G-modules over $G_{(1)}$. We use this, together with the computation of the varieties $V_{G_{(1)}}(H^0(\lambda))$, given by Jantzen (1987) in [8] and by Nakano et al. (2002) in [10], to explicitly compute the support variety of a block of $Dist(G_{(r)})$.

The aim of this paper is to provide computations of support varieties for modules over Frobenius kernels of algebraic groups. Specifically, for G a classical simple algebraic group over an algebraically closed field k of characteristic $p > 0$, we give a description (Theorem 3.2) of the support variety of a simple G-module over the r-th Frobenius kernel $G_{(r)}$ in terms of the support varieties of simple G-modules over $G_{(1)}$. Our proofs establish these results only under the assumption that p is large enough for the root system of G. A lower bound on p is provided in Section 3, roughly speaking it is the Coxeter number of G multiplied by a quadratic polynomial in the rank of G. In Section 4, we apply this result for $G = SL_n$ or Sp_{2n}, to give an explicit description of the support variety of a block of the distribution algebra $Dist(G_{(r)})$.

We should emphasize that the varieties computed in Section 3 can only be determined explicitly (by our results) if the support varieties of simple G-modules over $G_{(1)}$ are known explicitly, which is in general not the case. However, Drupieski et al. have in recent work [2] made such calculations for simple, simply-connected G, if one assumes that p is at least as large as the Coxeter number of G and that Lusztig's character formula holds for all restricted dominant weights.

The results in this paper rely most heavily on the work of Suslin et al. in [12,13]. In particular, all of our statements of support varieties are given in terms of varieties of 1-parameter subgroups, which the aforementioned papers prove to be homeomorphic to cohomologically defined support varieties. Moreover, the intuition behind our results for simple modules came from the calculations made in [13, 6.10] for Frobenius kernels of SL_2. We also use in an essential way the analysis and results of Carlson et al. in [1], and that of Friedlander in [3], both of which appear in the proof of Proposition 3.1. Finally, the results of Jantzen in [8], and the results and observations of Nakano et al. in [10] are critical to obtaining the calculations found in Section 4, where we compute the support variety of a block of $Dist(G_{(r)})$.

1. Preliminaries

We will assume throughout that k is an algebraically closed field of characteristic $p > 0$.

1.1. Representations of G

By a "classical" simple algebraic group, we shall mean that G is one of the groups SL_n, SO_n, or Sp_{2n} (thus excluding the simply-connected groups of types B and D). When viewing G as a subgroup of some GL_n, we will always assume this embedding is the "natural" one associated to G.

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Let T be a maximal split torus of G with character group $X(T)$, let Φ be the root system for G with respect to T, and fix a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. Denote by Φ^+ the set of positive roots with respect to Π, and let B^+ and B denote the Borel subgroups corresponding to Φ^+ and $-\Phi^+$, with their unipotent radicals denoted as U^+, U respectively. The Weyl group of Φ will be denoted W, and the dot action of $w \in W$ on $\lambda \in X(T)$ is defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half sum of the positive roots. We also denote by α_0 the highest short root.

We let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ for all roots α. The dominant integral weights of $X(T)$ are then given by

$$X(T)_+ := \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha_i^\vee \rangle, 1 \leq i \leq \ell\}.$$

The set of fundamental dominant weights, $\{\omega_1, \ldots, \omega_\ell\}$, is defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. For each $\lambda \in X(T)_+$, we denote by $L(\lambda)$ the unique simple G-module of highest weight λ. It is the socle of the induced module $\mathrm{H}^0(\lambda) := Ind_B^G(k_\lambda)$, where k_λ is the simple one-dimensional B-module of weight λ. The morphism $F : G \to G$ is the standard Frobenius morphism on GL_n restricted to G, and $G_{(r)} \subseteq G$ is the kernel of F^r. For a G-module M, we denote by $M^{(r)}$ the module which arises from pulling back M via F^r. The set of p^r-restricted weights of $X(T)$ is given by

$$X_r(T) := \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha_i^\vee \rangle < p^r, 1 \leq i \leq \ell\}.$$

As shown in [7, II.3], if $\lambda \in X_r(T)$, then $L(\lambda)$ remains simple upon restriction from G to $G_{(r)}$. Moreover, for G simply-connected, the set

$$\{L(\lambda) \mid \lambda \in X_r(T)\},$$

is a complete set of pairwise non-isomorphic simple $G_{(r)}$-modules.

1.2. Distribution algebras

If H is any affine group scheme, with coordinate algebra $k[H]$, and I_ϵ the augmentation ideal of $k[H]$, then the distribution algebra of H, $Dist(H)$, is defined by

$$Dist(H) = \{f \in \mathrm{Hom}_k(k[H], k) \mid f(I_\epsilon^n) = 0, \text{ for some } n \geq 1\}.$$

It follows that $Dist(H_{(r)}) \subseteq Dist(H_{(r+1)})$, and $Dist(H) = \bigcup_{r \geq 1} Dist(H_{(r)})$ (see [7, I.9] for more on Frobenius kernels of arbitrary affine group schemes). For a morphism of affine group schemes $\phi : H_1 \to H_2$, we denote by $d\phi : Dist(H_1) \to Dist(H_2)$ the induced map of algebras.

Of particular importance will be the structure of the algebra $Dist(\mathbb{G}_a)$. In this case, we have $k[\mathbb{G}_a] \cong k[t]$, and $Dist(\mathbb{G}_a)$ is spanned by the elements $(\frac{d}{dt})^{(j)}$, where

$$\left(\frac{d}{dt}\right)^{(j)} (t^i) = \delta_{ij}.$$

If we set $u_j = (\frac{d}{dt})^{(p^j)}$, and if m is an integer with p-adic expansion $m = m_0 + m_1 p + \cdots + m_q p^q$, then

$$\left(\frac{d}{dt}\right)^{(m)} = \frac{u_0^{m_0} \cdots u_q^{m_q}}{m_0! \cdots m_q!}.$$

Therefore $Dist(\mathbb{G}_a)$ is generated as an algebra over k by the set $\{u_j\}_{j \geq 0}$, while $Dist(\mathbb{G}_{a(r)})$ is generated by the subset where $j < r$.

With F^i denoting the i-th iterate of the Frobenius morphism as above, we have that the differential $dF^i : Dist(\mathbb{G}_a) \to Dist(\mathbb{G}_a)$ is given by

$$dF^i(u_j) = \begin{cases} u_{j-i} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

Let $\delta : \mathbb{G}_a \to \mathbb{G}_a \times \mathbb{G}_a$ be the morphism which sends g to (g, g), for all $g \in \mathbb{G}_a(A)$, and for all commutative k-algebras A. Then the differential of δ is the co-multiplication of $Dist(\mathbb{G}_a)$ (see [7, I.7.4]), so we will write $d\delta$ as $\Delta'_{\mathbb{G}_a}$. It is not hard to show that $\Delta'_{\mathbb{G}_a} : Dist(\mathbb{G}_a) \to Dist(\mathbb{G}_a) \otimes Dist(\mathbb{G}_a)$, is given by

$$\Delta'_{\mathbb{G}_a}\left(\left(\frac{d}{dt}\right)^{(n)}\right) = \sum_{i+j=n} \left(\frac{d}{dt}\right)^{(i)} \otimes \left(\frac{d}{dt}\right)^{(j)}.$$

1.3. Support varieties

We recall that the Frobenius kernel $H_{(r)}$ has finite dimensional coordinate algebra $k[H_{(r)}]$, and thus is a *finite group scheme*. By Friedlander and Suslin [5], we have then that the algebra