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Effective dimension of finite semigroups

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ABSTRACT

In this paper we discuss various aspects of the problem of determining the minimal dimension of an injective linear representation of a finite semigroup over a field. We outline some general techniques and results, and apply them to numerous examples.

1. Introduction

Most representation theoretic questions about a finite semigroup S over a field $\Bbbk$ are really questions about the semigroup algebra $\Bbbk S$. One question that is, however, strictly about S itself is the minimum dimension of an effective linear representation of S over $\Bbbk$, where by effective we mean injective; we call this the effective dimension of S over $\Bbbk$. Note that semigroups (and in fact groups) with isomorphic semigroup algebras can have different effective dimensions. For example, the effective dimension of $\mathbb{Z}/4\mathbb{Z}$ over $\mathbb{C}$ is 1, whereas the effective dimension of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is 2 (since $\mathbb{C}$ has a unique element of multiplicative order two), although both groups have algebras isomorphic to $\mathbb{C}^4$.

There are two natural questions that arise when considering the effective dimension of finite semigroups:

(a) Is the effective dimension of a finite semigroup decidable?
(b) Can one compute the effective dimension of one's favorite finite semigroups?

These are two fundamentally different questions. The first question asks for a Turing machine that on input the Cayley table of a finite semigroup, outputs the effective dimension over $\Bbbk$. The second one asks for an actual number. Usually for the second question one has in mind a family of finite semigroups given by some parameters, e.g., full (partial) transformation monoids, full linear monoids over finite fields, full monoids of binary relations, etc. One wants to know the effective dimension as a function of the parameters.

The effective dimension of groups (sometimes called the minimal faithful degree) is a classical topic, dating back to the origins of representation theory. There doesn't seem to be that much work in the literature on semigroups except for the paper [21] of Kim and Roush and previous work [27] of the authors. This could be due in part to the fact that the question is much trickier for semigroups because semigroup algebras are rarely semisimple. Also, minimal dimension effective modules need not be submodules of the regular representation.

Question (a) has a positive answer if the first order theory of the field $\Bbbk$ is decidable. Indeed, to determine the effective dimension of a finite semigroup one just needs to solve a finite collection of systems of equations and inequations over $\Bbbk$ because the effective dimension is obviously bounded by the size of the semigroup plus one. Classical results of Tarski imply

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that this is the case for algebraically closed fields and for real closed fields. However, the time complexity of these algorithms seems to be prohibitive to applying them in practice.

Question (b) is answered for all the classical finite semigroups mentioned above, as well as several other families.

Our main tools are the classical representation theory of finite semigroups (as in [13, Chapter 5], [35] and [18]), model theory, algebraic geometry and representation varieties, and George Bergman's lemma from [21]. We formulate various general techniques which can be used in the study of effective dimensions for certain classes of semigroups and along the way recover and improve upon many partial results in this direction that were known, at least on the level of folklore, to representation and semigroup theorists.

The paper is organized as follows. Section 2 discusses elementary properties of effective modules and effective dimension. In particular, the relevance of a classical result of Steinberg [43] is discussed. The main result in this section is an improvement on the obvious upper bound on effective dimension. The next section explains how to interpret the effective dimension of a finite semigroup over a field $\Bbbk$ in the first order theory of $\Bbbk$ and then applies model theoretic results to deduce a number of conclusions, including decidability over algebraically closed and real closed fields. Section 4 gives a simple description of the effective dimension of a commutative inverse monoid over the complex field (or any sufficiently nice field) using Pontryagin duality for finite commutative inverse monoids. These results in particular apply to finite abelian groups and to lattices, the former case of course being well known [20]. The following section studies the effective dimension of generalized group mapping semigroups in the sense of Krohn and Rhodes, see [24]. This class includes full partial transformations monoids, symmetric inverse monoids, full binary relation monoids and full linear monoids over finite fields. The effective dimension is computed in each of these cases.

Section 6 discusses a lemma from [21], which is attributed to G. Bergman. Kim and Roush had already used this lemma (and a variant) to compute the effective dimension of the semigroups of Hall relations and reflexive relations. The authors used it in previous work to compute the effective dimension of 0-Hecke monoids associated to finite Coxeter groups, see [27]. In Section 6, we use it to compute the effective dimension of semigroups of transformations with a doubly transitive group of units and at least one singular transformation. This applies in particular to full transformation monoids. The next section studies the effective dimension of nilpotent semigroups. Using elementary algebraic geometry and the notion of representation varieties, we show that generic n-dimensional representations of free nilpotent semigroups of nilpotency index n are effective over an algebraically closed field. It follows that the effective dimension of these semigroups is n. The same is true for free commutative nilpotent semigroups of index n. On the other hand, we construct arbitrarily large commutative nilpotent semigroups of any nilpotency index $n \geq 3$ with the property that effective dimension equals cardinality (this is the worse possible case). This leads one to guess that the computational complexity of computing effective dimension for nilpotent semigroups should already be high. Section 8 computes the effective dimension, over an algebraically closed field, of various types of path semigroups, including the path semigroup of an acyclic quiver and certain truncated path semigroups. Here, again, we use the technology of representation varieties.

The penultimate section considers some other examples that are essentially known in the literature, e.g., rectangular bands and symmetric groups, as well as some new results for hyperplane face semigroups and free left regular bands. In the last section we present a table of effective dimensions over the complex numbers of various classical families of finite semigroups.

2. Effective modules

Let $\Bbbk$ be a field, S a semigroup and V a vector space over $\Bbbk$. A linear representation $\varphi\colon S \to \mathrm{End}_{\Bbbk}(V)$ is said to be *effective* if it is injective. We shall also say that the module V is effective. If, furthermore, the linear extension $\overline{\varphi}\colon \Bbbk S \to \mathrm{End}_{\Bbbk}(V)$ is injective, we say that V is a *faithful* module. Our choice of terminology follows [17] and is aimed at avoiding confusion between these two different notions. Of course faithful modules are effective, but the converse is false. For example, for $n > 1$, the group $\mathbb{Z}/n\mathbb{Z}$ has an effective one-dimensional representation over $\mathbb{C}$ but no faithful one.

2.1. Steinberg's theorem

There is a well known result of Steinberg (see [43]) that says, in effect, that effective modules are not too far from faithful ones. The result was generalized by Rieffel to the context of bialgebras [36]. Recall that, given two S-modules V and W, the vector space $V \otimes W$ has the natural structure of an S-module given by

$$s(v \otimes w) := sv \otimes sw. \tag{1}$$

Technically speaking, $\Bbbk S$ is a bialgebra where the comultiplication $\triangle\colon \Bbbk S \to \Bbbk S \otimes \Bbbk S$ is given by $\triangle(s) = s \otimes s$ and the counit $\varepsilon\colon \Bbbk S \to \Bbbk$ given by $\varepsilon(s) = 1$ for all $s \in S$. For bialgebras one can always define the tensor product of representations.

Theorem 1 (*Steinberg*)**.** *Let V be an effective module for a semigroup S. Then*

$$\mathcal{T}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

is a faithful module.