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The cubic Hecke algebra on at most 5 strands

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In memory of Johann Gustav Hermes, who worked 10 years on completing the construction of the 65537-gon and on producing the corresponding beautiful artwork of drawings and numbers, nowadays known as 'Der Koffer' in Göttingen's library.

MSC: 20F36; 20C08

ABSTRACT

We prove that the quotient of the group algebra of the braid group on 5 strands by a generic cubic relation has finite rank. This was conjectured by Broué, Malle and Rouquier and has for consequence that this algebra is a flat deformation of the group algebra of the complex reflection group G_{32}, of order 155,520.

1. Introduction

In 1957 H.S.M. Coxeter proved (see [8]) that the quotient of the braid group B_n on $n \geq 2$ strands by the relations $s_i^k = 1$, where $s_1, \ldots, s_{n-1}$ denote the usual Artin generators, is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$. This means that, besides the obvious case $k = 2$, which leads to the symmetric group, and the case $n = 2$, there is only a finite number of such groups. They all turn out to be irreducible complex reflection groups, namely finite subgroups of $\mathrm{GL}_n(\mathbf{C})$ generated by endomorphisms which fix a hyperplane (so-called pseudo-reflections), and which leave no proper subspace invariant. In the usual classification of such objects, due to Shephard and Todd, they are nicknamed as G_4, G_8, G_{16} for $n = 3$ and $k = 3, 4, 5$, G_{25}, G_{32} for $n = 4, 5$ and $k = 3$.

In 1998, M. Broué, et al. conjectured (see [6]) that the group algebra of all complex reflection groups admit flat deformations similar to the Hecke algebra of a Weyl or Coxeter group. They actually introduced natural deformations of such group algebras, called them the (generic) Hecke algebra associated to such a group, and they conjectured that these were flat deformations, and in particular that they have finite rank. For the groups we are interested in, this conjecture actually amounts to saying that the quotients of the group algebra RB_n by the relations $s_i^k + a_{k-1}s_i^{k-1} + \cdots + a_1 s_i + a_0 = 0$, where $R = \mathbf{Z}[a_{k-1}, \ldots, a_1, a_0, a_0^{-1}]$, is a flat deformation of the group algebra RW, where $W = B_n/s_i^k$ (note that we actually use a slightly smaller ring than the one used in [6] and [5]). This conjecture was proved in [5] for G_4 and G_{25} but not for the largest cubic case G_{32} (Satz 4.7 – the proof for G_{25} is however only sketched there); actually, a preliminary version of the conjecture (where Hecke algebras were not associated to an arbitrary complex reflection group but instead to a specific kind of group presentation), already covering the cases that we are considering here, dates back to 1993, and is stated in [5] (Vermutung 4.6).

Note that, outside its original framework, the validity of this conjecture is assumed in a number of papers about so-called Cherednik algebras and related topics.

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According to [6] (see the proof of theorem 4.24 there) only the following needs to be proved: that the algebra is spanned over R by $|W|$ elements. This is what we prove here.

Theorem 1.1. *The generic Hecke algebra associated to $W = G_{32}$ is spanned by $|W|$ elements, and is thus a free R-module of rank $|W|$ which becomes isomorphic to the group algebra of W after a suitable extension of scalars.*

More precisely, according to [11] corollary 7.2, a convenient extension of scalars would be $\mathbf{Q}(\zeta_3, (\zeta_3^{-r}u_r)^{\frac{1}{6}}, r = 0, 1, 2)$ where ζ_3 is a primitive 3rd root of 1 and $X^3 + a_2X^2 + a_1X + a_0 = (X - u_0)(X - u_1)(X - u_2)$ or, better, the algebraic extension of $\mathbf{Q}(\zeta_3)(u_0, u_1, u_2)$ generated by $\sqrt{u_0u_1}$, $\sqrt{u_0u_2}$, $\sqrt{u_1u_2}$ and $\sqrt[3]{u_0u_1u_2}$ (see [11] table 8.2 and proposition 5.1).

In the general setting of complex reflection groups, it is known that this conjecture is true

- for the general series (usually denoted $G(de, e, r)$) of complex reflection groups (by works Ariki–Koike [2] and Ariki [1]),
- for most of the exceptional groups of rank 2 by [5] and [14], which are numbered G_4 to G_{22}. More precisely, only the groups G_{17}, G_{18} and G_{19} have not been checked yet. In [9], a weak version of the conjecture is proved for all exceptional groups of rank 2.
- for G_{25} by [5], for the groups G_{26}, G_{27} by computer means ([14]).
- for the Coxeter groups.

The remaining cases are in rank 4 the groups G_{29} ([14] however checked that the algebra has the right dimension over the field of fractions), G_{31}, G_{32}, in rank 5 the group G_{33} and in rank 6 the group G_{34}. All but G_{32}, whose case we settled here, have all their pseudo-reflections of order 2.

In the case studied here, we actually prove more. Here and in the sequel we denote A_n the quotient of RB_n by the generic cubic relation $s_i^3 - as_i^2 - bs_i - c = 0$. The usual embedding $B_n \hookrightarrow B_{n+1}$ induces a natural morphism $A_n \to A_{n+1}$, hence an A_n-bimodule structure on A_{n+1}. For $n \leq 4$, we give a decomposition of A_{n+1} as A_n-bimodule. This immediately provides an explicit R-basis of A_n for $n \leq 5$, made of images of braids in B_n. Recall that the orders of G_4, G_{25} and G_{32} are 24, 648 and 155,520.

The following theorem is a recollection of the main results of this article: see in particular Theorems 3.2, 4.1, 6.21 and 6.26 as well as Corollary 5.12, and recall that the argument of [6] theorem 4.24 (which involves a transcendental monodromy construction) shows that proving that the Hecke algebra of type W is R-generated by $|W|$ elements ensures that this Hecke algebra is free as an R-module, with basis the given $|W|$ elements. Moreover, notice that, if we have an inclusion of parabolic subgroups $W_0 \subset W$ with corresponding Hecke algebras $H_0 \subset H$, knowing the conjecture for H_0 and that H is generated by $|W/W_0|$ elements as an H_0-module proves (1) the conjecture for H and (2) that H is free as an H_0-module, with basis these elements. Indeed, letting $N = |W/W_0|$ the assumption provides an H_0-module morphism $H_0^N \to H$; composing with $(R^{|W_0|})^N \simeq H_0^N$ this yields a surjective morphism $R^{|W|} \to H$ which is an isomorphism by the argument of [6]. This proves that the original morphism $H_0^N \to H$ has no kernel either, and so is an isomorphism.

Theorem 1.2.
- *Let $S_2 = \{1, s_1, s_1^{-1}\} \subset B_2$. One has $|S_2| = 3$ and S_2 provides an R-basis of A_2.*
- *Let $S_3 = S_2 \sqcup S_2s_2^{\pm}S_2 \sqcup S_2s_2^{-1}s_1s_2^{-1} \subset B_3$. One has $|S_3| = 24$ and S_3 provides an R-basis of A_3.*
- *A_4 is a free A_3-module of rank 27. A basis of this A_3-module is provided by elements of the braid group (including 1) which map to a system of representatives of G_{25}/G_4.*
- *A_4 is a free R-module of rank 648. A basis of this R-module is provided by elements of the braid group including 1 which map to all G_{25}.*
- *A_4 is a free $A_2 \otimes_R A_2 \simeq \langle s_1, s_3\rangle$-module of rank 72. A basis of this $\langle s_1, s_3\rangle$-module is provided by elements of the braid group including 1 which map to a system of representatives of $G_{25}/(\mathbf{Z}/3\mathbf{Z})^2$.*
- *A_5 is a free A_4-module of rank 240. A basis is provided by elements of the braid group including 1 which map to a system of representatives of G_{32}/G_{25}.*
- *A_5 is a free R-module of rank 155,520. A basis of this R-module is provided by elements of the braid group which include 1 and which map to all G_{32}.*

Corollary 1.3. *The natural map $A_n \to A_{n+1}$ is injective for $2 \leq n \leq 4$.*

We describe the plan of the proof. Our method is inductive. We find generators of A_{n+1} as an A_n-bimodule, and only then as an A_n-module. After some preliminaries in Section 2 we do the case of A_3 in Section 3. The structure of A_4 as an A_3-module is obtained in Section 4. Before considering A_5, we provide in Section 5 an alternative description of A_4, this time as a $\langle s_1, s_3\rangle$-module. In addition to providing an alternative proof of the conjecture for A_4, this is used in the decomposition of A_5 as an A_4-module. This decomposition is obtained in Section 6. We first obtain a decomposition of A_5 as an A_4-bimodule, and introduce a filtration of A_5 by simpler A_4-bimodules. The latest step of the filtration has original generators originating from the center of the braid group, and this turns out to be the crucial reason why this filtration terminates, thus proving that A_5 is an R-module of finite rank. For proving this crucial property one needs a lengthy calculation which is postponed in Section 7. We conclude the Section 6 and the proof of the main theorem by studying the structure as A_4-modules of the A_4-bimodules involved there.

Remark 1.4. A detailed version of this paper, with more computations detailed, can be found on the arxiv. For publication purposes, we skip here the details for quite a few computations. In particular, we assert without proof the equalities between words in s_1, s_2, s_3, s_4 when they are (sometimes not so easy) identities inside the braid group. By using normal forms for elements in the braid group, the verification of such identities can be easily automatized.