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Comparison of admissibility conditions for cyclotomic Birman–Wenzl–Murakami algebras

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ABSTRACT

We show the equivalence of admissibility conditions proposed by Wilcox and Yu (in press) [11] and by Rui and Xu (2009) [9] for the parameters of cyclotomic BMW algebras.

1. Introduction

Cyclotomic Birman–Wenzl–Murakami (BMW) algebras are BMW analogues of cyclotomic Hecke algebras [2,1]. They were defined by Häring–Oldenburg in [7] and have recently been studied by three groups of mathematicians: Goodman and Hauschild-Mosley [4–6,3], Rui, Xu, and Si [9,8], and Wilcox and Yu [11,12,10,13].

A peculiar feature of these algebras is that it is necessary to impose "admissibility" conditions on the parameters entering into the definition of the algebras in order to obtain a satisfactory theory. There is no one obvious best set of conditions, and the different groups studying these algebras have proposed different admissibility conditions and have chosen slightly different settings for their work.

Under their various admissibility hypotheses on the ground ring, the several groups of mathematicians mentioned above have obtained similar results for the cyclotomic BMW algebras, namely freeness and cellularity. In addition, Goodman & Hauschild-Mosley and Wilcox & Yu have shown that the algebras can be realized as algebras of tangles, while Rui et al. have obtained additional representation theoretic results, for example, classification of simple modules and semisimplicity criteria. However, it has been difficult to compare the results of the different investigations because of the different settings.

The purpose of this note is to show that the admissibility condition proposed by Rui and Xu [9] is equivalent to the condition proposed by Wilcox and Yu [11]. As a result, one has a consensus setting for the study of cyclotomic BMW algebras.

Further background on cyclotomic BMW algebras, motivation for the study of these algebras, relations to other mathematical topics (quantum groups, knot theory), and further literature citations can be found in [5] and in the other papers cited above.

2. Definitions

In general we use the definitions and notation from [6].

Definition 2.1. Fix an integer $r \geq 1$. A *ground ring* S is a commutative unital ring with parameters ρ, q, δ_j ($j \geq 0$), and $u_1, \ldots, u_r$, with ρ, q, and $u_1, \ldots, u_r$ invertible, and with

$$\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1). \tag{2.1}$$

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Definition 2.2. Let S be a ground ring with parameters ρ, q, δ_j ($j \geq 0$), and $u_1, \ldots, u_r$. The *cyclotomic BMW algebra* $W_{n,S,r}(u_1, \ldots, u_r)$ is the unital S-algebra with generators $y_1^{\pm 1}$, $g_i^{\pm 1}$ and e_i ($1 \leq i \leq n-1$) and relations:

(1) (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $y_1 y_1^{-1} = y_1^{-1} y_1 = 1$.
(2) (Idempotent relation) $e_i^2 = \delta_0 e_i$.
(3) (Affine braid relations)
  (a) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ and $g_i g_j = g_j g_i$ if $|i-j| \geq 2$.
  (b) $y_1 g_1 y_1 g_1 = g_1 y_1 g_1 y_1$ and $y_1 g_j = g_j y_1$ if $j \geq 2$.
(4) (Commutation relations)
  (a) $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.
  (b) $y_1 e_j = e_j y_1$ if $j \geq 2$.
(5) (Affine tangle relations)
  (a) $e_i e_{i\pm 1} e_i = e_i$,
  (b) $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$ and $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$.
  (c) For $j \geq 1$, $e_1 y_1^j e_1 = \delta_j e_1$.
(6) (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
(7) (Untwisting relations) $g_i e_i = e_i g_i = \rho^{-1} e_i$ and $e_i g_{i\pm 1} e_i = \rho e_i$.
(8) (Unwrapping relation) $e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1$.
(9) (Cyclotomic relation) $(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0$.

Thus, a cyclotomic BMW algebra is the quotient of the affine BMW algebra [7,4], by the cyclotomic relation $(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0$. We recall from [4] that the affine BMW algebra is isomorphic to an algebra of framed affine tangles, modulo Kauffman skein relations. Assuming admissible parameters, it has been shown that the cyclotomic BMW algebras are also isomorphic to tangle algebras [6,10,13].

Lemma 2.3. *For $j \geq 1$, there exist elements $\delta_{-j} \in \mathbb{Z}[\rho^{\pm 1}, q^{\pm 1}, \delta_0, \ldots, \delta_j]$ such that $e_1 y_1^{-j} e_1 = \delta_{-j} e_1$. Moreover, the elements δ_{-j} are determined by the recursion relation:*

$$\begin{aligned} \delta_{-1} &= \rho^{-2}\delta_1 \\ \delta_{-j} &= \rho^{-2}\delta_j + (q^{-1} - q)\rho^{-1} \sum_{k=1}^{j-1} (\delta_k \delta_{k-j} - \delta_{2k-j}) \quad (j \geq 2). \end{aligned} \tag{2.2}$$

Proof. Follows from [4], Corollary 3.13, and [5], Lemma 2.6; or [9], Lemma 2.17. □

We consider what are the appropriate morphisms between ground rings for cyclotomic BMW algebras. The obvious notion would be that of a ring homomorphism taking parameters to parameters; that is, if S is a ground ring with parameters ρ, q, etc., and S' another ground ring with parameters ρ', q', etc., then a morphism $\varphi : S \to S'$ would be required to map $\rho \mapsto \rho'$, $q \mapsto q'$, etc.

However, it is better to require less, for the following reason: The parameter q enters into the cyclotomic BMW relations only in the expression $q^{-1} - q$, and the transformation $q \mapsto -q^{-1}$ leaves this expression invariant. Moreover, the transformation $g_i \mapsto -g_i$, $\rho \mapsto -\rho$, $q \mapsto -q$ (with all other generators and parameters unchanged) leaves the cyclotomic BMW relations unchanged.

Taking this into account, we arrive at the following notion:

Definition 2.4. Let S be a ground ring with parameters ρ, q, δ_j ($j \geq 0$), and $u_1, \ldots, u_r$. Let S' be another ground ring with parameters ρ', q', etc.

A unital ring homomorphism $\varphi : S \to S'$ is a *morphism of ground rings* if it maps

$$\begin{cases} \rho \mapsto \rho', \text{ and} \\ q \mapsto q' \text{ or } q \mapsto -q'^{-1}, \end{cases}$$

or

$$\begin{cases} \rho \mapsto -\rho', \text{ and} \\ q \mapsto -q' \text{ or } q \mapsto q'^{-1}, \end{cases}$$

and strictly preserves all other parameters.

Suppose there is a morphism of ground rings $\psi : S \to S'$. Then ψ extends to a homomorphism from $W_{n,S,r}$ to $W_{n,S',r}$. Moreover, $W_{n,S,r} \otimes_S S' \cong W_{n,S',r}$ as S'-algebras. These statements are discussed in [6], Section 2.4.