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Motivic cohomology of the simplicial motive of a Rost variety

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ABSTRACT

We compute the motivic cohomology groups of the simplicial motive $\mathcal{X}_\theta$ of a Rost variety for an n-symbol θ in Galois cohomology of a field. As an application we compute the kernel and cokernel of multiplication by θ in Galois cohomology. We also show that the reduced norm map on K_2 of a division algebra of square-free degree is injective.

1. Motivic cohomology of $\mathcal{X}_\theta$

1.1. Introduction

Let l be a prime integer, F a field of characteristic different from l. The Galois cohomology group $H^1_{et}(F, \mu_l)$, where μ_l is the Galois module of all lth roots of unity, is canonically isomorphic to the factor group $F^\times/F^{\times l}$. We write (a) for the class in $H^1_{et}(F, \mu_l)$ corresponding to an element $a \in F^\times$. Let $a_1, \ldots, a_{n-1} \in F^\times$ for some $n \geq 1$ and $\chi \in H^1_{et}(F, \mathbb{Z}/l\mathbb{Z})$. We consider the n-tuple of 1-dimensional cohomology classes

$$\theta = \big(\chi, (a_1), \ldots, (a_{n-1})\big).$$

Abusing notation we shall also write θ for the cup-product $\chi \cup (a_1) \cup \cdots \cup (a_{n-1})$ in $H^n_{et}(F, \mu_l^{\otimes(n-1)})$ and call this element a *symbol*.

Note that if $\mu_l \subset F^\times$, the choice of a primitive lth root of unity identifies $\mathbb{Z}/l\mathbb{Z}$ with μ_l and, therefore, χ with (a_0) for some $a_0 \in F^\times$. Thus, θ is given by the n-tuple $(a_0, a_1, \ldots, a_{n-1})$ of elements in $F^\times$.

A *Rost variety* for θ is a smooth projective variety X_θ over F satisfying the conditions given in [20, Def. 1.1] or [3, Def. 0.5].

Example 1.1 (*See [20]*). (1) If $n = 1$, then $X_\theta = \mathrm{Spec}(L)$, where L/F is a cyclic field extension of degree l splitting θ, is a Rost variety for θ.

(2) If $n = 2$, the Severi–Brauer variety $X_\theta = SB(A)$ of a central simple F-algebra A of dimension l^2 with the class θ in $H^2(F, \mu_l) \subset \mathrm{Br}(F)$ is a Rost variety for θ.

An inductive process given in [13] allows us to construct a Rost variety for any θ. Denote further by $\mathcal{X}_\theta$ the Čech simplicial scheme $\check{C}(\mathcal{X}_\theta)$ of X_θ (see [17, Appendix B]) and by $M(\mathcal{X}_\theta)$ the motive of $\mathcal{X}_\theta$ in the triangulated category $\mathbf{DM}(F, \mathbb{Z})$ (see [6]).

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The motive of $\mathcal{X}_\theta$ in $\mathbf{DM}(F, \mathbb{Z}_{(l)})$ is independent of the choice of the Rost variety X_θ ([16, Section 5]). If $\theta = 0$, then $\mathcal{X}_\theta = \mathbb{Z}$, so in general, $\mathcal{X}_\theta$ is a "twisted form" of $\mathbb{Z}$. We write $H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ for the motivic cohomology group $H^{p,q}\big(M(\mathcal{X}_\theta), \mathbb{Z}\big)$.

The triviality of the motivic cohomology group $H^{n+1,n}(\mathcal{X}_\theta, \mathbb{Z})$ is the essential step in the proof of Bloch–Kato Conjecture (see [16, Prop. 6.11]). In this paper we compute the motivic cohomology $H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ for all p and q (Theorem 1.15).

In the second part of the paper some applications are given. We compute the kernel and cokernel of multiplication by θ in Galois cohomology. We also show that the reduced norm map on K_2 of a division algebra of square-free degree is injective.

We use the following notation:
$K_*(F)$ is the Milnor ring of a field F.
If X is a variety over F, we write $A_0(X, K_p)$ for the cokernel of the residue homomorphism (see [11]):

$$\coprod_{x\in X_{(1)}} K_{p+1}F(x) \to \coprod_{x\in X_{(0)}} K_pF(x),$$

where $X_{(i)}$ is the set of all points of X of dimension i.

$n \geq 2$ an integer,
$b = (l^{n-1} - 1)/(l - 1) = 1 + l + \cdots + l^{n-2}$,
$c = (l^n - 1)/(l - 1) = 1 + l + \cdots + l^{n-1} = bl + 1 = b + l^{n-1}$,
$d = l^{n-1} - 1 = b(l - 1) = c - b - 1$.

1.2. The Bloch–Kato Conjecture and the motivic cohomology of $\mathcal{X}_\theta$

The Bloch–Kato Conjecture asserts that the norm residue homomorphism

$$h_{n,l} : K_n(F)/lK_n(F) \to H^n_{et}(F, \mu_l^{\otimes n}),$$

taking the class of a symbol $\{a_0, a_1, \ldots, a_{n-1}\}$ to the cup-product $(a_0)\cup(a_1)\cup\cdots\cup(a_{n-1})$, is an isomorphism. This conjecture was proved in [16] (see also [3,13,19–21]). In view of [14], the natural maps

$$H^{p,q}(Y, \mathbb{Z}) \to H^{p,q}_{et}(Y, \mathbb{Z})$$

are isomorphisms for a smooth projective variety Y over F and $p \leq q + 1$. Moreover, the natural map

$$H^{p,q}(\mathcal{X}_\theta, \mathbb{Z}) \to H^{p,q}_{et}(\mathcal{X}_\theta, \mathbb{Z}) \tag{1}$$

is an isomorphism if $p \leq q + 1$. By [17, Lemma 7.3],

$$H^{p,q}_{et}(\mathcal{X}_\theta, \mathbb{Z}) \simeq H^{p,q}_{et}(F, \mathbb{Z}) \tag{2}$$

for all p and q.

For every $\mathcal{N} \in \mathbf{DM}(F, \mathbb{Z})$ and every $\alpha \in H^{p,q}(\mathcal{N}, \mathbb{Z})$ the *order of* α is the integer $\operatorname{ord}(\alpha) = p - q - 1$. The subgroup of $H^{*,*}(\mathcal{N}, \mathbb{Z})$ of elements of non-negative (respectively, non-positive) order will be denoted by $H^{*,*}(\mathcal{N}, \mathbb{Z})^{\geq 0}$ (respectively, $H^{*,*}(\mathcal{N}, \mathbb{Z})^{\leq 0}$).

1.3. The motive $\widetilde{\mathcal{X}}_\theta$

The motive $\widetilde{\mathcal{X}}_\theta$ is defined by the exact triangle

$$\widetilde{\mathcal{X}}_\theta \to M(\mathcal{X}_\theta) \to \mathbb{Z} \to \widetilde{\mathcal{X}}_\theta[1] \tag{3}$$

in $\mathbf{DM}(F, \mathbb{Z})$. Note that the motive $\widetilde{\mathcal{X}}_\theta$ differs by a shift from the one defined in [17].

It follows from (1) and (2) that

$$H^{p,q}(\mathcal{X}_\theta, \mathbb{Z}) \simeq H^{p,q}_{et}(\mathcal{X}_\theta, \mathbb{Z}) \simeq H^{p,q}_{et}(F, \mathbb{Z}) \simeq H^{p,q}(F, \mathbb{Z}) \tag{4}$$

if $p \leq q + 1$. As $H^{p,q}(F, \mathbb{Z}) = 0$ when $p > q$, the exact triangle (3) yields:

Proposition 1.2. *There are canonical isomorphisms:*

$$H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})^{\geq 0} \simeq H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0},$$
$$H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\leq 0} \simeq H^{*,*}(F, \mathbb{Z})^{\leq 0},$$
$$H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})^{\leq 0} = 0.$$