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# On varieties of almost minimal degree III: Tangent spaces and embedding scrolls

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ABSTRACT

Let $X \subset \mathbb{P}^r$ be a variety of almost minimal degree which is the projected image of a rational normal scroll $\tilde{X} \subset \mathbb{P}^{r+1}$ from a point $p$ outside of $\tilde{X}$. In this paper we study the tangent spaces at singular points of $X$ and the geometry of the embedding scrolls of $X$, i.e. the rational normal scrolls $Y \subset \mathbb{P}^r$ which contain $X$ as a codimension one subvariety.



## 1. Introduction

Varieties of minimal degree, namely (irreducible non-degenerate) projective varieties $X \subseteq \mathbb{P}^r$ with $\deg(X) = \operatorname{codim}(X) + 1$ have been studied and classified already since the 19th century by del Pezzo in the case of surfaces and by Bertini in the general case. Varieties of almost minimal degree, e.g. projective varieties $X \subseteq \mathbb{P}^r$ which satisfy the equality $\deg(X) = \operatorname{codim}(X) + 2$ are still an active branch of projective algebraic geometry. These latter varieties were studied and classified by Fujita (see [9] or [11]). A purely algebraic approach to varieties of almost minimal degree was given by Hoa et al. [13] in 1991. In [3] it was shown that varieties $X \subseteq \mathbb{P}^r$ of almost minimal degree which are either non-linearly normal or non-normal are precisely the linear projections of varieties $\tilde{X} \subseteq \mathbb{P}^{r+1}$ of minimal degree from a point $p \in \mathbb{P}^{r+1} \backslash \tilde{X}$.

So, understanding varieties of almost minimal degree which are either non-linearly normal or else non-normal is equivalent to knowing the possible linear projections $\pi_p : \tilde{X} \longrightarrow X_p := \pi_p(\tilde{X})$ of a variety of minimal degree $\tilde{X} \subseteq \mathbb{P}^{r+1}$ from points $p \in \mathbb{P}^{r+1} \backslash \tilde{X}$. If $\tilde{X}$ is (a cone over) the Veronese surface in $\mathbb{P}^5$, this is a task which can be solved easily. In the "general case", namely if the projecting variety $\tilde{X} \subseteq \mathbb{P}^{r+1}$ is a (cone over) a smooth rational normal scroll this same task turns out to be more demanding. The crucial point here consist in knowing how the so called secant locus

$$\Sigma_p(\tilde{X}) := \{q \in \tilde{X} \mid \#(\langle p, q\rangle \cap \tilde{X}) > 1\}$$

of $\tilde{X}$ with respect to the center of projection $p$ depends on $p$. In [2] we have solved this problem, by making explicit the so called secant stratification of $\tilde{X}$. One application of this is an extension of Fujita's classification of normal del Pezzo varieties to possibly non-normal del Pezzo varieties (see [2]).

In the present paper, we are concerned with local aspects of varieties of almost minimal degree.

Our first aim is to determine the embedding dimension $\dim(T_x X)$ and the multiplicity $m_x(X)$ of a closed singular point $x$ of a variety $X \subseteq \mathbb{P}^r$ of almost minimal degree which is not normal. It turns out that for all such points $x$ which are not vertex points of $X$ we have

$$\dim(T_x X) = 2\dim(X) + 2 - \operatorname{depth}(X) \quad \text{and} \quad m_x(X) = 2,$$

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where depth($X$) denotes the arithmetic depth of $X$ (see Theorem 3.9). Clearly the behavior of the tangent spaces $T_xX$ of a variety $X \subseteq \mathbb{P}^r$ of almost minimal degree is closely related to the question how the tangent spaces $T_{q_1}\tilde{X}, T_{q_2}\tilde{X}$ of the projecting variety $\tilde{X} \subseteq \mathbb{P}^{r+1}$ in two distinct points $q_1, q_2 \in \tilde{X}$ intersect. Again, the case where $\tilde{X} \subseteq \mathbb{P}^{r+1}$ is a rational normal scroll is crucial here. We treat this problem completely in the case where $\tilde{X}$ is smooth (see Theorem 4.2) and a cone (see Corollary 4.4). Once more, the secant stratification of $\tilde{X}$ is the basic tool we need to do this.

The final Sections 5 and 6 are devoted to the study of the so called embedding scrolls $Y \subseteq \mathbb{P}^r$ of a variety $X \subseteq \mathbb{P}^r$ of almost minimal degree $\geq 5$, that is of scrolls $Y$ containing $X$ and satisfying $\dim(Y) = \dim(X) + 1$. In [3] it is shown that these embedding scrolls always exist. They are a very useful tools for the study of Betti diagrams of varieties of almost minimal degree (see [3,15] and in particular [16]). Our aim is to give an account of all possible embedding scrolls of a given variety $X \subseteq \mathbb{P}^r$ of almost minimal degree (which is not a cone). We show that the singular embedding scrolls of $X$ are always of the shape $Y = \text{Join}(\text{Sing}(X), X)$, and hence unique, and that in the case where $2 \leq \text{depth}(X) \leq \dim(X)$ there are no smooth embedding scrolls (see Theorems 5.5 and 5.8). We also describe the possible smooth embedding scrolls in the case $\text{depth}(X) = \dim(X) + 1$, that is if $X$ is maximally del Pezzo. In this situation we have (see Theorem 6.10)

- a one-dimensional family of smooth embedding scrolls if $X$ is a curve
- a unique smooth embedding scroll if the projecting scroll $\tilde{X}$ is a surface without line sections
- no smooth embedding scroll in the remaining cases.

In the case where $X$ is smooth, we show that all its embedding scrolls are smooth (see Theorem 5.5(a)). But in this case, we are not able yet to describe all possible embedding scrolls (which indeed occur in families now).

## 2. Preliminaries

**Notation and Remark 2.1.** (A) Let $K$ be an algebraically closed field, let $r$ be an integer $\geq 2$ and let $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ be a variety of minimal degree with

$$n := \dim(\tilde{X}) \quad \text{and} \quad e := \text{codim}(\tilde{X}) \geq 2.$$

So, $\tilde{X} \subseteq \mathbb{P}_K^{r+1}$ is either a rational normal scroll or (a cone over) the Veronese surface in $\mathbb{P}_K^5$. Keep in mind that $\tilde{X}$ is integral, non-degenerate, arithmetically normal, arithmetically Cohen–Macaulay (CM) and of degree $e + 1$.

(B) Now, let $p \in \mathbb{P}_K^{r+1} \setminus \tilde{X}$ be a closed point. We fix a projective space $\mathbb{P}_K^r$ and a linear projection

$$\pi_p : \tilde{X} \to X := \pi_p(\tilde{X}) \subseteq \mathbb{P}_K^r$$

of $\tilde{X}$ from $p$. We may consider $\mathbb{P}_K^r$ as a subspace of $\mathbb{P}_K^{r+1}$ with $p \notin \mathbb{P}_K^r$ and $\pi_p$ as given by the canonical projection of $\mathbb{P}_K^{r+1}$ from $p$ onto $\mathbb{P}_K^r$, so that $\pi_p^{-1}(x) = \tilde{X} \cap \langle x, p \rangle$ for all closed points $x \in X$. Keep in mind that $\pi_p$ is finite and birational and that $X$ is *a variety of almost minimal degree*, in the sense of [3], so that $\deg(X) = \text{codim}(X) + 2$.

(C) Keep the above notations and consider the *secant cone of* $\tilde{X}$ with respect to $p$, defined by

$$\text{Sec}_p(\tilde{X}) := \bigcup_{q \in \tilde{X} : \text{length}(\tilde{X} \cap \langle p, q \rangle) \geq 2} \langle p, q \rangle$$

if $\tilde{X}$ admits secant lines passing through $p$, and $\text{Sec}_p(\tilde{X}) = \{p\}$ else. We furnish $\text{Sec}_p(\tilde{X})$ with its reduced scheme structure. We also introduce the *secant locus* of $\tilde{X}$ with respect to $p$, which is defined as the scheme theoretic intersection

$$\Sigma_p(\tilde{X}) := \text{Sec}_p(\tilde{X}) \cap \tilde{X}.$$

Let us also consider the *arithmetic depth* of $X$, which we denote by $t$, thus

$$t := \text{depth}(X).$$

In these notations we have (see [3, Theorem 1.3]):

(2.1) *If $t = 1$, then $\tilde{X}$ and $X$ are smooth, $\pi_p : \tilde{X} \to X$ is an isomorphism and $X$ is not linearly normal.*

(2.2) *If $t \geq 2$, then $\text{Sec}_p(\tilde{X}) = \mathbb{P}_K^{t-1} \subseteq \mathbb{P}_K^{r+1}$, the secant locus $\Sigma_p(\tilde{X}) \subseteq \mathbb{P}_K^{t-1}$ is a hyperquadric and $\pi_p(\Sigma_p(\tilde{X})) = \mathbb{P}_K^{t-2}$ is the non-normal locus of $X$.*

In addition, if $X$ is not arithmetically CM, then the generic point of the non CM-locus of $X$ is of Goto type. More precisely:

(2.3) *If $1 \leq t \leq n$, then $\pi_p(\Sigma_p(\tilde{X}))$ is the non CM-locus of $X$ and the generic point $x$ of this locus satisfies*

$$H^i_{\mathfrak{m}_{X,x}}(\mathcal{O}_{X,x}) \cong \begin{cases} 0, & \text{if } i \neq 1, \dim(\mathcal{O}_{X,x}), \\ \kappa(x), & \text{if } i = 1. \end{cases}$$

(D) According to [3] a *maximal del Pezzo* variety $X \subseteq \mathbb{P}_K^r$ is a variety of almost minimal degree which is arithmetically CM. These are indeed the del Pezzo varieties in the sense of Fujita [11,10] which are in addition linearly normal. A *del Pezzo variety* is a projective variety which is the image of a maximal del Pezzo variety under a linear isomorphic projection. Using this terminology we can say (see [3, Theorem 1.2], [9,11]):