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Descent in $*$-autonomous categories

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ABSTRACT

We extend the result of Joyal and Tierney asserting that a morphism of commutative algebras in the $*$-autonomous category of sup-lattices is an effective descent morphism for modules if and only if it is pure, to an arbitrary $*$-autonomous category $\mathcal{V}$ (in which the tensor unit is projective) by showing that any $\mathcal{V}$-functor out of $\mathcal{V}$ is precomonadic if and only if it is comonadic.

1. Introduction

General descent theory, as originally developed by Grothendieck [5] in the abstract setting of fibred categories, is an invaluable tool in algebraic geometry, which one can also apply to various situations in Galois theory, topology and topos theory. A general aim of descent theory is to give characterizations of the so-called (effective) descent morphisms, which in the case of a fibred category satisfying the Beck–Chevalley condition reduces to monadicity of a suitable functor.

A basic example of Grothendieck's descent theory involves modules over commutative rings. Consider a homomorphism of commutative rings $i : A \to B$ and the corresponding *extension-of-scalars* functor $i_* = B \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_B$. It is well known that this functor admits as a right adjoint the underlying functor $i^* : \mathrm{Mod}_B \to \mathrm{Mod}_A$. The problem of Grothendieck's descent theory for modules is concerned with the characterization of those B-modules M for which there exists an A-module N and an isomorphisms $i_*(N) \simeq M$ of B-modules. To be more specific, let M be a B-module and let $\theta_M : M \otimes_A B \to B \otimes_A M$ be a homomorphism of $B \otimes_A B$-modules, where $B \otimes_A B$ acts on $M \otimes_A B$ by $(b_1 \otimes b_2)(m \otimes b) = b_1 m \otimes b_2 b$ and on $B \otimes_A M$ by $(b_1 \otimes b_2)(b \otimes m) = b_1 b \otimes b_2 m$. Define

$$(\theta_M)_1 : B \otimes_A M \otimes_A B \to B \otimes_A B \otimes_A M,$$
$$(\theta_M)_2 : M \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A M,$$
$$(\theta_M)_3 : M \otimes_A B \otimes_A B \to B \otimes_A M \otimes_A B$$

by tensoring θ_M with 1_B in the first, second and third positions respectively. *Descent data* on a B-module M is an isomorphism $\theta_M : M \otimes_A B \to B \otimes_A M$ of $B \otimes_A B$-modules such that $(\theta_M)_2 = (\theta_M)_1 \cdot (\theta_M)_3$.Des$(i)$ denotes the category of pairs (M, θ_M), θ_M descent data on a B-module M, where morphisms $(M, \theta_M) \to (M', \theta_{M'})$ are morphisms $f : M \to M'$ of B-modules that commute with descent data in the obvious way. For any A-module N, there is an isomorphism $\theta_{i_*(N)} : N \otimes_A B \otimes_A B \to B \otimes_A N \otimes_A B$, arising from

$$(i_1)_*(i_*(N)) = (i_1 i)_*(N) = (i_2 i)(N) = (i_2)_*(i_*(N)),$$

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where $i_1, i_2 : B \to B \otimes_A B$ are the maps defined by $i_1(b) = 1 \otimes_A b$ and $i_2(b) = b \otimes_A 1$. Explicitly $\theta_{i_*(N)}$ is the map given by

$$n \otimes_A b_1 \otimes_A b_2 \longmapsto b_1 \otimes_A n \otimes_A b_2.$$

Thus one has a functor

$$K_i : \mathrm{Mod}_A \to \mathrm{Des}(i).$$

One says that $i : A \to B$ is an (effective) descent homomorphism of commutative rings if the functor K_i is full and faithful (an equivalence of categories). So that, when i is an effective descent morphism, to specify an A-module is to specify a B-module M together with descent data $\theta_M : M \otimes_A B \to B \otimes_A M$ of $B \otimes_A B$-modules. The descent theory for modules is thus the study of which homomorphisms of commutative rings $i : A \to B$ are (effective) descent morphisms. Grothendieck [5] proved that faithfully flat extensions of commutative rings are effective. A full characterization of effective descent morphisms for modules was given by A. Joyal and M. Tierney (unpublished, but see [12]) and by Olivier [13]: a morphism $i : A \to B$ of commutative rings is an effective descent morphism iff i is a pure morphism of A-modules. For example, if i is a split monomorphism of A-modules, then it is effective for descent (see, for example, [6]).

According to the theorem of Bénabou and Roubaud [1] and J. Beck (unpublished), the category $\mathrm{Des}(i)$ is equivalent to the Eilenberg–Moore category of $\mathbf{G}_i$-coalgebras, where $\mathbf{G}_i$ is the comonad on the category Mod_B generated by the adjunction $i_* \dashv i^* : \mathrm{Mod}_B \to \mathrm{Mod}_A$. Modulo this equivalence, the functor $K_i : \mathrm{Mod}_A \to \mathrm{Des}(i)$ can be identified with the comparison functor $K_{\mathbf{G}_i} : \mathrm{Mod}_A \to (\mathrm{Mod}_B)_{\mathbf{G}_i}$, and thus to say that i is an (effective) descent morphism is to say that the extension-of-scalars functor $i_* : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is precomonadic (comonadic).

Since purity of any homomorphism of commutative rings is equivalent to precomonadicity of the corresponding extension-of-scalars functor, Grothendieck's descent theory for modules over commutative rings can be conceived by interpreting a descent result as a statement asserting that for a given homomorphism of commutative rings, precomonadicity of the corresponding extension-of-scalars functor implies (and hence is equivalent to) comonadicity.

Identifying the homomorphism $i : A \to B$ with the algebra B in the monoidal category Mod_A and considering the monad $\mathbf{T}_i$ on Mod_A given by tensoring with B, the category Mod_B can be seen as the Eilenberg–Moore category of $\mathbf{T}_i$-algebras and the functor i_* as the comparison functor $K_{\mathbf{T}_i} : \mathrm{Mod}_A \to (\mathrm{Mod}_A)^{\mathbf{T}_i}$. Thus the problem of effectiveness of i is equivalent to the one of the comonadicity of the functor i_*. This motivates to call a monad $\mathbf{T}$ on a category $\mathscr{A}$ to be of (effective) descent type if the free $\mathbf{T}$-algebra functor $F^{\mathbf{T}} : \mathscr{A} \to \mathscr{A}^{\mathbf{T}}$ is precomonadic (comonadic). Hence, in the language of monads, the Joyal–Tierney theorem can be paraphrased as follows: For any pure homomorphism of commutative rings $i : A \to B$, the monad $\mathbf{T}_i$ is an effective descent morphism iff it is of descent type.

The question whether a given morphism is effective for descent has been investigated for various categories. An important example of wide applicability of Grothendieck's descent theory is Joyal–Tierney's result [8] on a descent theory for open maps of locales. In [8], A. Joyal and M. Tierney looked at descent theory in the context of the $*$-autonomous category $\mathcal{CL}$ of sup-lattices and indicated that there was a useful analogy with the descent theory for modules over commutative rings. A main result of [8] asserts that a morphism of commutative algebras in $\mathcal{CL}$ is effective for descent iff it is pure. This is used to show that open surjections are effective in the category of locales, leading to a representation theorem for Grothendieck topoi (over a base topos) in terms of localic groupoids, which can be considered as a generalization of the fundamental theorem of Galois theory.

Our aim is to generalize the descent theorem of Joyal–Tierney for sup-lattices by showing that for any $*$-autonomous category $\mathcal{V}$ with an injective dualizing object, a $\mathcal{V}$-functor is precomonadic iff it is comonadic.

The paper is organized as follows. Section 2 rather technical, and is devoted to extend classical monadicity results to the enriched setting. Section 3 contains some criteria for comonadicity that are used in the next section to generalize the theorem of Joyal–Tierney to $*$-autonomous categories.

2. Preliminaries

We let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ denote a symmetric monoidal category, where $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$ is the tensor product of $\mathcal{V}$ and where I is the tensor unit. A $\mathcal{V}$-functor will be called a functor if it is understood that the domain and codomain are $\mathcal{V}$-categories. Similarly a $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors will be called simply a natural transformation. For $\mathcal{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, $[\mathscr{A}, \mathscr{B}]_0$ will denote the ordinary category of functors from $\mathscr{A}$ to $\mathscr{B}$ and natural transformations between them. In our paper we follow the notation from [9], which is our general reference for enriched category theory.

We start by recalling the basic facts about $\mathcal{V}$-monads; all can be found in [3,4].

Given $\mathcal{V}$-category $\mathscr{A}$, a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathscr{A}$ consists of a functor $T : \mathscr{A} \to \mathscr{A}$ together with natural transformations $\eta : 1_{\mathscr{A}} \to T$ and $\mu : T^2 \to T$ satisfying the usual three axioms.

As in the ordinary case, every $\mathcal{V}$-adjunction $\eta, \epsilon : F \dashv U : \mathscr{B} \to \mathscr{A}$ induces a monad on $\mathscr{A}$ by letting $\mathbf{T} = (UF, \eta, U\epsilon F)$.

Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on a $\mathcal{V}$-category $\mathscr{A}$. Then clearly $\mathbf{T}_0 = (T_0, \eta_0, \mu_0)$ is an ordinary monad on $\mathscr{A}_0$, and one has the Eilenberg–Moore category $\mathscr{A}_0^{\mathbf{T}_0}$ and the free-forgetful adjunction

$$F^{\mathbf{T}_0} \dashv U^{\mathbf{T}_0} : \mathscr{A}_0^{\mathbf{T}_0} \to \mathscr{A}_0$$

determining the monad $\mathbf{T}_0$.