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Pointfree forms of Dowker's and Michael's insertion theorems

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ABSTRACT

In this paper we prove two strict insertion theorems for frame homomorphisms. When applied to the frame of all open subsets of a topological space they are equivalent to the insertion statements of the classical theorems of Dowker and Michael regarding, respectively, normal countably paracompact spaces and perfectly normal spaces. In addition, a study of perfect normality for frames is made.

1. Introduction

Theorems on the existence of continuous real functions on normal spaces rank among the fundamental results in point-set topology. They can, for instance, be divided into three groups: separation theorems (such as Urysohn's lemma), extension theorems (such as Tietze's theorem), and insertion theorems. The latter theorems are the strongest ones in the sense that they yield the former as very easy corollaries. It is therefore of importance to consider them in the more general setting of pointfree topology. This paper is a sequel to the authors' earlier papers regarding pointfree insertion (see [25,17,14,15]). For the reader's convenience we first record the three basic insertion theorems of Katětov–Tong [19,29], Dowker [5] and Michael [24].

Theorem A (*Katětov–Tong*). *A topological space X is normal if and only if, given $h, g : X \to \mathbb{R}$ such that $h \le g$, h is upper semicontinuous and g is lower semicontinuous, there is a continuous $f : X \to \mathbb{R}$ such that $h \le f \le g$.*

Theorem B (*Dowker*). *A topological space X is normal and countably paracompact if and only if, given $h, g : X \to \mathbb{R}$ such that $h < g$, h is upper semicontinuous and g is lower semicontinuous, there is a continuous $f : X \to \mathbb{R}$ such that $h < f < g$.*

Theorem C (*Michael*). *A topological space X is perfectly normal if and only if, given $h, g : X \to \mathbb{R}$ such that $h \le g$, h is upper semicontinuous and g is lower semicontinuous, there is a continuous $f : X \to \mathbb{R}$ such that $h \le f \le g$ and $h(x) < f(x) < g(x)$ whenever $h(x) < g(x)$.*

In pointfree setting, Theorem A was first investigated by Li and Wang [23] with, however, some discrepancy between topological and frame semicontinuities. Right frame semicontinuities and right pointfree version of Theorem A have been fixed by Picado [25] and Gutiérrez García and Picado [17].

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In this paper, we aim to provide some forms of Theorems B and C for, respectively, normal countably paracompact frames and perfectly normal frames. In the pointfree setting the situation becomes much more complex than in the topological case and we have not been able to provide pointfree assertions corresponding exactly to the insertion statements of Theorems B and C. For instance, in both cases we assume $h = \mathbf{0}$. It should however be emphasized that both Theorems B and C easily follow from their pointfree versions established in this paper. These versions are corollaries of a rather general insertion lemma related to an arbitrary frame L with a certain extra order $\Subset$ which in turn is an abstract version of a result of Gutiérrez García and Kubiak [13] concerning a normal topology $\mathcal{O}X$ with $U \Subset V$ iff $\mathrm{int}(X \setminus U) \cup V = X$. We also establish some natural results regarding perfectly normal frames. These include separation and extension theorems for perfectly normal spaces. We have not been able to deduce them from our pointfree Michael's theorem. These are deduced from our general insertion lemma.

2. Background in frames

I. Frames and locales. The category Frm of *frames* has as objects those complete lattices L in which

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$$

for all $a \in L$ and $B \subseteq L$. Morphisms, called *frame homomorphisms*, are those maps between frames that preserve arbitrary joins (hence 1, the top) and finite meets (hence 0, the bottom). The set of all morphisms from L into M is denoted by $\mathsf{Frm}(L, M)$. The category of locales is the opposite category of Frm.

Motivating example: the lattice $\mathcal{O}X$ of all open subsets of a space X is a frame and if $f : X \to Y$ is a map, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism.

II. Heyting operator. With L a frame and $a \in L$, the map $a \wedge (\cdot) : L \to L$ preserves arbitrary joins and so has a right adjoint $a \to (\cdot) : L \to L$ determined by $c \le a \to b$ iff $a \wedge c \le b$. Thus, $a \to b = \bigvee \{c \in L : a \wedge c \le b\}$. For all $a, b, c \in L$ and $B \subseteq L$ the following hold:

(H1) $a \to b = a \to (a \wedge b)$,
(H2) $a \wedge b = a \wedge c$ iff $a \to b = a \to c$,
(H3) $a \to \bigwedge B = \bigwedge_{b \in B}(a \to b)$.

The *pseudocomplement* of $a \in L$ is $a^* = a \to 0$. Clearly, $a \wedge a^* = 0$.

III. Sublocales. An $S \subseteq L$ is a *sublocale* of L if, given $A \subseteq S$ and $a \in L$, one has $\bigwedge A \in S$ and $a \to s \in S$ for all $s \in S$ (see [18, p. 50] and [26]). Each sublocale $S \subseteq L$ is itself a frame with $\wedge$ and $\to$ of L (the top of S is 1, while the bottom 0_S of S may differ from 0). It determines the surjection (frame quotient) $c_S : L \to S$ given by $c_S(x) = \bigwedge \{s \in S : x \le s\}$. The sublocales of L form a complete lattice $(\mathcal{S}(L), \subseteq)$ with $\{1\}$ being the bottom $\mathsf{0}$, L being the top $\mathsf{1}$, and in which, given $\{S_j : j \in J\} \subseteq \mathcal{S}(L)$, one has

$$\bigwedge_{j \in J} S_j = \bigcap_{j \in J} S_j \quad \text{and} \quad \bigvee_{j \in J} S_j = \left\{ \bigwedge A : A \subseteq \bigcup_{j \in J} S_j \right\}.$$

Then $\mathcal{S}(L)$ is a co-frame.

For any $a \in L$, the sets

$$\mathfrak{o}(a) = \{a \to b : b \in L\} \quad \text{and} \quad \mathfrak{c}(a) = \uparrow a$$

are sublocales of L called, respectively, *open* and *closed*. Clearly, the quotients $c_{\mathfrak{o}(a)}$ and $c_{\mathfrak{c}(a)}$ are given by

$$c_{\mathfrak{o}(a)}(x) = a \to x \quad \text{and} \quad c_{\mathfrak{c}(a)}(x) = a \vee x.$$

Properties 2.1. *We shall freely use the following properties:*

(1) $\mathfrak{o}(a) \subseteq \mathfrak{o}(b)$ *iff* $a \le b$,
(2) $\mathfrak{o}(\bigvee A) = \bigvee_{a \in A} \mathfrak{o}(a)$,
(3) $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ *iff* $a \vee b = 1$ *iff* $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$,
(4) $\mathfrak{o}(a)$ *and* $\mathfrak{c}(a)$ *are complements of each other.*

IV. The frame of reals. Being algebraic, the category Frm allows definitions by generators and relations. Using this, one can constructively define the frame of reals in terms of $\mathbb{Q}$ [9]. Following the more recent detailed description in [2], the *frame of reals* $\mathfrak{L}(\mathbb{R})$ is one generated by $\mathbb{Q} \times \mathbb{Q}$ satisfying the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \le r < q \le s$,
(R3) $(p, q) = \bigvee \{(r, s) : p < r < s < q\}$,
(R4) $1 = \bigvee_{p, q \in \mathbb{Q}} (p, q)$.

We write: $(p, -) = \bigvee_{q > p} (p, q)$ and $(-, q) = \bigvee_{p < q} (p, q)$.

A morphism having $\mathfrak{L}(\mathbb{R})$ as a domain will be defined on the sets of their generators. Such a map uniquely determines a frame homomorphism if and only if it turns the relations holding for generators into identities (see [2] for details).