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Projective modules over smooth, affine varieties over Archimedean real closed fields

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ABSTRACT

Let $X = \operatorname{Spec}(A)$ be a smooth, affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers. Let P be a projective A-module of rank n such that its nth Chern class $C_n(P) \in \mathrm{CH}_0(X)$ is zero. In this set-up, Bhatwadekar–Das–Mandal showed (amongst many other results) that $P \simeq A \oplus Q$ in the case that either n is odd or the topological space $X(\mathbb{R})$ of real points of X does not have a compact, connected component. In this paper, we prove that similar results hold for smooth, affine varieties over an Archimedean real closed field $\mathbf{R}$.

1. Introduction

Let $X = \operatorname{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a field k and let P be a projective A-module of rank n. It is well known that in general P may not split off a free summand of rank one. Hence, it is of interest to find sufficient conditions for this to happen. When k is an algebraically closed field, a result of Murthy [1, Theorem 3.8] says that if the top Chern class $C_n(P)$ in $\mathrm{CH}_0(X)$ is zero, then P splits off a free summand of rank one (i.e. $P \simeq A \oplus Q$). Note that over any base field, the vanishing of the top Chern class is a necessary condition for P to split off a free summand of rank one. However, the example of the tangent bundle of an even dimensional sphere shows that this condition is not sufficient. Therefore, it is natural to ask: *under what further conditions* $C_n(P) = 0 \overset{?}{\Rightarrow} P \simeq A \oplus Q$. In the case $k = \mathbb{R}$, this question was initially investigated in [2] and brought to a satisfactory conclusion in [3], e.g. it has been shown (amongst many other results) in [3, Theorem 4.30] that when n is odd, then $C_n(P) = 0$ implies that $P \simeq A \oplus Q$. Moreover it is also shown that in the case n is even, $\wedge^n(P) \not\simeq K_A$, then $C_n(P) = 0$ is a sufficient condition for P to have a free summand of rank one, where K_A denotes the canonical module of A over $\mathbb{R}$. In this paper we extend these results to the case when the base field k is an Archimedean real closed field. More precisely, we prove:

Theorem 1.1. *Let $\mathbf{R}$ be an Archimedean real closed field. Let $X = \operatorname{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over $\mathbf{R}$. Let $X(\mathbf{R})$ denote the $\mathbf{R}$-rational points of the variety. Let K denote the canonical module $\wedge^n(\Omega^*_{A/\mathbf{R}})$. Let P be a projective A-module of rank n and let $\wedge^n(P) = L$. Assume that $C_n(P) = 0$ in $\mathrm{CH}_0(X)$. Then $P \simeq A \oplus Q$ in the following cases:*

1. *$X(\mathbf{R})$ has no closed and bounded semi-algebraically connected component.*
2. *For every closed and bounded semi-algebraically connected component W of $X(\mathbf{R})$, $L_W \not\simeq K_W$ where K_W and L_W denote restriction of (induced) line bundles on $X(\mathbf{R})$ to W.*
3. *n is odd.*

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Moreover, if n is even and L is a rank 1 projective A-module such that there exists a closed and bounded semi-algebraically connected component W of $X(\mathbf{R})$ with the property that $L_W \simeq K_W$, then there exists a projective A-module P of rank n such that $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ (hence $C_n(P) = 0$) but P does not have a free summand of rank 1.

Note that when the base field is $\mathbb{R}$, the semi-algebraically connected semi-algebraic components are the connected components of $X(\mathbb{R})$ in the Euclidean topology.

We thank the referee for pointing out an error in the proof of (3.1) in an earlier version and suggesting a way to correct it.

2. Preliminaries

The first part can be looked upon as a quick reference guide to the theory of real closed fields and the topological notions related to them. More details can be found in [4].

Definition 2.1. A field $\mathbf{R}$ is said to be real if it can be ordered in a way such that addition and multiplication are compatible with the ordering. An equivalent definition is that $\sum_{i=1}^{n} a_i^2 = 0 \Rightarrow a_i = 0 \forall i$. A real closed field is a real field which has no algebraic extensions which are real, equivalently attaching a root of -1 makes it algebraically closed.

Such fields come with a natural topology based on intervals like in the case of $\mathbb{R}$. However, under this topology, the field itself is not connected (except in the case of $\mathbb{R}$).

Definition 2.2. A subset V of $\mathbf{R}^n$ is called a basic semi-algebraic set if V is of the form

$$\{x \in \mathbf{R}^n \mid f_i(x) = 0, g_j(x) > 0, 1 \le i \le r, 1 \le j \le s\},$$

where $f_i(x), g_j(x) \in \mathbf{R}[X_1, X_2, \ldots, X_n]$. A subset W of $\mathbf{R}^n$ is called a semi-algebraic set if W is a finite union of basic semi-algebraic sets.

A semi-algebraic subset W of $\mathbf{R}^n$ is semi-algebraically connected if for every pair of disjoint, closed, semi-algebraic subsets F_1 and F_2 of W, satisfying $F_1 \cup F_2 = W$, either $F_1 = W$ or $F_2 = W$.

Now we quote a result, the proof of which can be found in [4, Theorem 2.4.4].

Theorem 2.3. *Every semi-algebraic subset W of $\mathbf{R}^l$ is the disjoint union of a finite number of semi-algebraically connected semi-algebraic subsets $W_1, W_2, \ldots, W_s$ which are closed in W. The $W_1, W_2, \ldots, W_s$ are called the* ***semi-algebraically connected semi-algebraic components*** *of W.*

Remark 2.4. When the field is $\mathbb{R}$, the semi-algebraically connected semi-algebraic components are same as the connected components by [4, Theorem 2.4.5].

Let $\mathbf{R} \hookrightarrow \mathbf{R}'$ be real closed fields. Let $X = \text{Spec}(A)$ be a smooth affine variety over $\mathbf{R}$ and let $X(\mathbf{R})$ denote the set of $\mathbf{R}$-rational points of X. Let $A' = A \otimes_{\mathbf{R}} \mathbf{R}'$ and let $X' = \text{Spec}(A')$ be the corresponding (smooth) affine variety over $\mathbf{R}'$. Note that, fixing a closed embedding of $X(\mathbf{R})$ in $\mathbf{R}^l$ (for suitable l), we can regard the topological space $X(\mathbf{R})$ as a subspace of $X(\mathbf{R}')$. Let $W_1', W_2', \ldots, W_s'$ be the semi-algebraically connected semi-algebraic components of $X(\mathbf{R}')$. Let $W_i = W_i' \cap X(\mathbf{R})$. Then, $W_1, W_2, \ldots, W_s$ are precisely the semi-algebraically connected semi-algebraic components of $X(\mathbf{R})$ (for a proof of a more general result see [4, Proposition 5.3.6]). Note that W_i' is closed and bounded if and only if W_i is closed and bounded.

Now we state the Artin–Lang homomorphism theorem [4, Thm. 4.1.2].

Theorem 2.5. *Let A be a finite type $\mathbf{R}$-algebra. If there exists an $\mathbf{R}$-algebra homomorphism $\phi : A \to \mathbf{R}'$ into a real closed extension $\mathbf{R}'$ of $\mathbf{R}$, then there exists an $\mathbf{R}$-algebra homomorphism $\psi : A \to \mathbf{R}$.*

In particular, if A is an $\mathbf{R}$-subalgebra of $\mathbf{R}'$, then we get a retraction from A to $\mathbf{R}$.

To make the paper self-contained, we define the Euler Class Group. Once again, more details can be obtained in either [3] or [5].

Definition 2.6. Definition of $\text{E}(A, L)$ and $\text{E}_0(A, L)$

Let A be a ring of dimension $n \ge 2$ and let L be a projective A-module of rank 1. Write $F = L \oplus A^{n-1}$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Two surjections α, β from F/JF to J/J^2 are said to be related if there exists $\sigma \in SL_{A/J}(F/JF)$ such that $\alpha\sigma = \beta$. Clearly this is an equivalence relation on the set of surjections from F/JF to J/J^2. Let $[\alpha]$ denote the equivalence class of α. Such an equivalence class $[\alpha]$ is called a *local L-orientation* of J. By abuse of notation, we shall identify an equivalence class $[\alpha]$ with α. A local L-orientation α is called a *global L-orientation* if $\alpha : F/JF \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : F \twoheadrightarrow J$.

Let G be the free abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$ where $\mathcal{N}$ is an $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements and $\omega_{\mathcal{N}}$ is a local L-orientation of $\mathcal{N}$. Now let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements and ω_J be a local L-orientation of J. Let $J = \cap_i \mathcal{N}_i$ be the (irredundant)