Journal of Pure and Applied Algebra 214 (2010) 689–700

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Relations between the Clausen and Kazhdan–Lusztig representations of the symmetric group

Charles Buehrle, Mark Skandera *

Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA

ARTICLE INFO

Article history:
Received 19 January 2008
Received in revised form 18 June 2009
Available online 6 August 2009
Communicated by R. Vakil

MSC:
20C30
20C08
15A15
17B37

ABSTRACT

We use Kazhdan–Lusztig polynomials and subspaces of the polynomial ring $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ to give a new construction of the Kazhdan–Lusztig representations of $\mathfrak{S}_n$. This construction produces exactly the same modules as those which Clausen constructed using a different basis in [M. Clausen, Multivariate polynomials, standard tableaux, and representations of symmetric groups, J. Symbolic Comput. (11), 5-6 (1991) 483–522. Invariant-theoretic algorithms in geometry (Minneapolis, MN, 1987)], and does not employ the Kazhdan–Lusztig preorders. We show that the two resulting matrix representations are related by a unitriangular transition matrix. This provides a $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$-analog of results due to Garsia and McLarnan, and McDonough and Pallikaros, who related the Kazhdan–Lusztig representations to Young's natural representations.

1. Introduction

In 1979, Kazhdan and Lusztig [1] introduced a family of irreducible modules for Coxeter groups and related Hecke algebras. The defining bases of these modules and corresponding matrix representations have many fascinating properties. Important steps in the construction of the Kazhdan–Lusztig modules are the computation of certain polynomials in $\mathbb{N}[q]$ known as *Kazhdan–Lusztig polynomials*, and the description of preorders on Coxeter group elements known as the *Kazhdan–Lusztig preorders*. These two tasks have become interesting research topics in their own right. For even the simplest case of a Coxeter group and corresponding Hecke algebra, the symmetric group $\mathfrak{S}_n$ and type-A Hecke algebra $H_n(q)$, the Kazhdan–Lusztig polynomials and preorders are somewhat poorly understood. (See, e.g., [2,3] and the references listed there.) These difficulties have led authors to study irreducible $\mathfrak{S}_n$-representations indexed by partitions λ of n and to search for a connection between the matrices $\{X_1^{\lambda}(w) \mid w \in \mathfrak{S}_n\}$ of the Kazhdan–Lusztig representations and those of other more elementary representations.

One well-known family of elementary $\mathfrak{S}_n$-representations is that of Young's *natural* representations. (See [4].) The traditional construction of natural representations employs a module defined in terms of a basis of combinatorial objects called *polytabloids*. A second family of elementary $\mathfrak{S}_n$-representations is that of Clausen's *bideterminant* representations [5]. Clausen defined these in terms of subspaces of the polynomial ring $\mathbb{C}[x] = \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ and bases of polynomials called *bideterminants* which had appeared earlier in the work of Mead [6], Désarménien–Kung–Rota [7], and others.

Garsia and McLarnan [8] and McDonough and Pallikaros [9] described the connection between matrices $\{X_2^{\lambda}(w) \mid w \in \mathfrak{S}_n\}$ of each natural representation and those of the corresponding Kazhdan–Lusztig representation as conjugation by a unitriangular matrix $B = B(\lambda)$,

$$X_1^{\lambda}(w) = B^{-1} X_2^{\lambda}(w) B, \quad \text{for all } w \in \mathfrak{S}_n. \tag{1.1}$$

* Corresponding address: Department of Mathematics, Lehigh University, 14 East Packer Avenue, Christmas-Saucon Hall, Bethlehem, PA 18015, United States.

E-mail addresses: ceb405@lehigh.edu (C. Buehrle), mas906@lehigh.edu, mark.skandera@gmail.com (M. Skandera).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2009.07.014

The former authors used properties of the Kazhdan–Lusztig and natural modules to solve Eq. (1.1) for B. The latter authors proved (1.1) by considering a third family of $\mathfrak{S}_n$-modules which are subspaces of $\mathbb{C}\mathfrak{S}_n$. (More precisely, they worked with representations and subspaces of $H_n(q)$.) Specifically, they showed that each such module has a polytabloid-inspired basis which yields Young's natural representation and a Kazhdan–Lusztig-inspired basis which yields the Kazhdan–Lusztig representation. Thus B is a transition matrix which relates the two bases. Moreover, this alternative construction of the Kazhdan–Lusztig representations does not rely upon preorders. (See also [10, Rmk. 2.3(i)], [11, Sec. 5] for related earlier constructions of preorder-avoiding modules.)

Proving results analogous to those above, we will describe the connection between the matrices $\{X_3^\lambda(w) \mid w \in \mathfrak{S}_n\}$ of each bideterminant representation and those of the corresponding Kazhdan–Lusztig representation as conjugation by a unitriangular matrix. We will accomplish this by giving a new construction of the Kazhdan–Lusztig representations. Specifically, we will use the second author's formulation [12, Thm. 2.1] of the *dual canonical basis* of $\mathbb{C}[x]$ to define a second basis of Clausen's bideterminant module, and will show that this basis produces the Kazhdan–Lusztig representations. Thus our unitriangular matrix $A = A(\lambda)$ defined by the equations

$$X_1^\lambda(w) = A^{-1}X_3^\lambda(w)A, \quad \text{for all } w \in \mathfrak{S}_n \tag{1.2}$$

is a transition matrix relating the bideterminant and dual canonical bases of the bideterminant module. Like the McDonough–Pallikaros construction, our new construction of the Kazhdan–Lusztig representations does not rely upon preorders. (See also [13] for an earlier appearance of the transition matrix A, and [14], [15] for previous related work on the dual canonical basis.)

In Sections 2–3, we review basic definitions related to the symmetric group, Hecke algebra, and Kazhdan–Lusztig modules. In Section 4 we review definitions related to the polynomial ring $\mathbb{C}[x]$ and a particular $n!$-dimensional subspace of $\mathbb{C}[x]$ called the *immanant space*. We recall the definition of the bideterminant basis of the immanant space and Clausen's use of this basis to construct irreducible $\mathfrak{S}_n$-modules [5]. In Section 5 we review basic definitions related to a noncommutative analog $\mathcal{A}(x; q)$ of $\mathbb{C}[x]$, and a certain *immanant subspace* of this ring. We then use the basis of Kazhdan–Lusztig immanants introduced in [14] to give a new construction of the Kazhdan–Lusztig representations of $H_n(q)$. Our modules are quotients of the immanant space of $\mathcal{A}(x; q)$, and like the original Kazhdan–Lusztig modules, they rely upon the Kazhdan–Lusztig preorders.

In Section 6, we specialize our new $H_n(q)$-modules at $q^{\frac{1}{2}} = 1$ to obtain $\mathfrak{S}_n$-modules which are subspaces of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$. Borrowing ideas from Clausen, and applying vanishing properties of Kazhdan–Lusztig immanants obtained in [13], we then modify our specialized modules to eliminate all quotients. This leads to our main result that this last family of $\mathfrak{S}_n$-modules gives a new, preorder-free construction of the Kazhdan–Lusztig representations of $\mathfrak{S}_n$. We finish by showing that the relationship between the bideterminant and Kazhdan–Lusztig immanant bases studied in [13, Sec. 5] leads to unitriangular transition matrices relating Clausen's irreducible representations of $\mathfrak{S}_n$ to those of Kazhdan and Lusztig.

2. The symmetric group, tableaux, and partial orders

The standard presentation of the symmetric group $\mathfrak{S}_n$ is given by generators $s_1, \ldots, s_{n-1}$ and relations

$$\begin{aligned} &s_i^2 = 1, \quad \text{for } i = 1, \ldots, n-1, \\ &s_is_js_i = s_js_is_j, \quad \text{if } |i-j| = 1, \\ &s_is_j = s_js_i, \quad \text{if } |i-j| \geq 2. \end{aligned} \tag{2.1}$$

We let $\mathfrak{S}_n$ act on rearrangements of the letters $[n] = \{1, \ldots, n\}$ by

$$s_i \circ v_1 \cdots v_n \underset{\text{def}}{=} v_1 \cdots v_{i-1}v_{i+1}v_iv_{i+2} \cdots v_n, \tag{2.2}$$

and we define the *one-line notation* of a permutation $w = s_{i_1} \cdots s_{i_\ell} \in \mathfrak{S}_n$ by

$$w_1 \cdots w_n \underset{\text{def}}{=} s_{i_1} \circ (\cdots (s_{i_\ell} \circ (1 \cdots n)) \cdots). \tag{2.3}$$

It is well known that this one-line notation does not depend upon the particular expression $s_{i_1} \cdots s_{i_\ell}$ for w. We say that such an expression is *reduced* if ℓ is as small as possible. We then call $\ell = \ell(w)$ the *length* of w.

We define the *Bruhat order* on $\mathfrak{S}_n$ by $v \leq w$ if some (equivalently every) reduced expression for w contains a reduced expression for v as a subword. (See [16] for more information). We call a generator s a *left ascent* for a permutation v if we have $sv > v$, and a *left descent* otherwise. Right ascents and descents are defined analogously. We denote the unique maximal element in the Bruhat order by w_0. This permutation has one-line notation $n(n-1)\cdots 21$. It is well known that the maps $v \mapsto w_0vw_0$ and $v \mapsto v^{-1}$ induce automorphisms of the Bruhat order, while the maps $v \mapsto vw_0$ and $v \mapsto w_0v$ induce antiautomorphisms. Thus we have

$$v \leq w \Leftrightarrow v^{-1} \leq w^{-1} \Leftrightarrow w_0vw_0 \leq w_0ww_0 \Leftrightarrow ww_0 \leq vw_0 \Leftrightarrow w_0w \leq w_0v. \tag{2.4}$$

A vector space V on which $\mathfrak{S}_n$ acts as a group of linear transformations is called an $\mathfrak{S}_n$-*module*. Any fixed basis for V then yields a *representation* of $\mathfrak{S}_n$ as a group of matrices. Important $\mathfrak{S}_n$-modules and $\mathfrak{S}_n$-representations termed *irreducible* are indexed by weakly decreasing sequences $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers which sum to n. (See, e.g., [4].) We call such