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# Explicit $K_2$ of some finite group rings

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**Abstract**

We compute $K_2$ of some finite group algebras of characteristic 2, giving explicit Steinberg or Dennis–Stein symbols as generators. The groups include finite abelian groups of 4-rank at most 1, some direct products involving semidirect product groups, and some small alternating groups.


*MSC:* Primary: 19C20; secondary: 19D50; 20C05; 16S34; 16W25

## 0. Introduction

The $K_2$ of a ring is the Schur multiplier of the group of elementary matrices over the ring. This work parallels the computations of $K_1$ in the author's paper "Explicit $K_1$ of some modular group rings" (see [9]). We find explicit symbols generating the $K_2$ of some finite group algebras using group rings over noncommutative coefficients, an exact sequence of Vorst and Weibel, and hyperelementary induction for $K_2(\mathbb{F}[-])$. One consequence is the nonvanishing of $K_2(\mathbb{Z}G)$ for the alternating groups $G$ of degree 4 and 5. Some computations are given for higher $K$-groups of finite group rings as well.

## 1. Notation and basic facts

Suppose $R$ is a ring (with unit). Let $\mathcal{P}(R)$ denote the category of finitely generated projective $R$-modules. In [14], Quillen defines $K_n(R)$ to be the homotopy group $\pi_{n+1}(BQ\mathcal{P}(R), 0)$, where $BQ\mathcal{P}(R)$ is the classifying space of a category $Q\mathcal{P}(R)$ constructed from $\mathcal{P}(R)$. Any exact functor from $\mathcal{P}(R)$ to $\mathcal{P}(S)$ induces a homomorphism of groups from $K_n(R)$ to $K_n(S)$; naturally isomorphic functors induce the same homomorphism.

This definition coincides with the "classical" definitions of $K_n(R)$ for $n = 0, 1, 2$. For $n = 0$, $K_0(R)$ is the Grothendieck group of $\mathcal{P}(R)$. Bass defined $K_1(R)$ to be the abelianization $GL(R)^{ab}$ of the infinite dimensional general linear group $GL(R)$. In [10], Milnor defined $K_2(R)$ as follows.

The group $E(R) = [GL(R), GL(R)]$ is generated by matrices $e_{ij}(r)$ (for $i \neq j$), which differ from the identity in having $r$ in the $i, j$-entry. The Steinberg group $St(R)$ has generators called $x_{ij}(r)$ (for $i \neq j, r \in R$) and defining

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relations:

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s),$$
$$[x_{ij}(r), x_{kl}(s)] = 1 \quad \text{if } i \neq l,\ j \neq k,$$
$$[x_{ij}(r), x_{jl}(s)] = x_{il}(rs) \quad \text{if } i \neq l.$$

These are modelled on relations among the $e_{ij}(r)$, so that replacing $x$ by $e$ defines a surjective group homomorphism $\phi$ from $St(R)$ to $E(R)$. Then $K_2(R)$ is the kernel of $\phi$ (and the center of $St(R)$). So $K_2(R)$ measures the non-generic relations among the matrices $e_{ij}(r)$. Further, the extension

$$0 \longrightarrow K_2(R) \longrightarrow St(R) \xrightarrow{\phi} E(R) \longrightarrow 0$$

is an initial object in the category of central extensions of $E(R)$; so $K_2(R) \cong H_2(E(R), \mathbb{Z})$.

To describe elements of $K_2(R)$, consider first some elements of $St(R)$ which $\phi$ takes to some simple matrices in $E(R)$. If $u \in R^*$,

$$w_{12}(u) = x_{12}(u)x_{21}(-u^{-1})x_{12}(u),$$
$$h_{12}(u) = w_{12}(u)w_{12}(-1).$$

If $u$ and $v$ are commuting units of $R$, the Steinberg symbol

$$\{u, v\} = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}$$

belongs to $K_2(R)$. It is multiplicative in each argument, antisymmetric, and $\{u, v\} = 1$ whenever $u + v$ is 1 or 0 in $R$. If $R$ is a commutative semilocal ring, $K_2(R)$ is generated by its Steinberg symbols (see [3], Theorem 2.7).

If $f : R \to S$ is a ring homomorphism, there is a group homomorphism $St(f)$ from $St(R)$ to $St(S)$ taking $x_{ij}(r)$ to $x_{ij}(f(r))$. This restricts to a group homomorphism $K_2(f)$ from $K_2(R)$ to $K_2(S)$ with

$$K_2(f)(\{u, v\}) = \{f(u), f(v)\}$$

for commuting units $u$ and $v$ of $R$. The Quillen and classical functors $K_2$ from rings to abelian groups are naturally isomorphic (see [15], Corollary 2.6 and Theorem 5.1).

## 2. Product decompositions

If $R$ and $S$ are rings, the projection maps induce an isomorphism

$$K_n(R \times S) \cong K_n(R) \times K_n(S)$$

for all $n \geq 0$ by [14], Section 2. If $m \geq 1$, tensoring with the $R, M_m(R)$-bimodule $R^m$ is an exact functor and category equivalence from $\mathcal{P}(M_m(R))$ to $\mathcal{P}(R)$, inducing an isomorphism $K_n(M_m(R)) \cong K_n(R)$. If $\mathbb{F}_q$ is a finite field with $q$ elements, Quillen proved that $K_n(\mathbb{F}_q)$ is 0 if $n > 0$ is even, and is cyclic of order $q^{(n+1)/2} - 1$ if $n$ is odd (see [13], Theorem 8).

If $f : R \to S$ is a surjective homomorphism of commutative semilocal rings, Bass showed that $R^* \to S^*$ is surjective (in [1], Chapter III, Corollary 2.9); so Steinberg symbols lift, and $K_2(f) : K_2(R) \to K_2(S)$ is surjective. In [4], Corollary 4.4(a), Dennis and Stein proved that $K_2(\mathbb{F}_q[x]/(x^m)) = 0$ for all $m \geq 1$. And in [2], Dennis, Keating and Stein showed that

$$K_2(\mathbb{F}_q[\mathbb{Z}_p^r]) \cong \mathbb{Z}_p^{f(r-1)(p^r-1)}$$

if $p$ is prime and $q = p^f$.

Equipped with these tools we can readily observe:

**Theorem 1.** *If $\mathbb{F}$ is a finite field of characteristic $p$ and $G$ is a finite group whose Sylow $p$-subgroup is a cyclic direct factor, then $K_2(\mathbb{F}G) = 0$. For a finite abelian group $G$, $K_2(\mathbb{F}G) = 0$ if and only if the Sylow $p$-subgroup of $G$ is cyclic.*