Journal of Pure and Applied Algebra 214 (2010) 53–60

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Trivial extensions defined by Prüfer conditions

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ARTICLE INFO

Article history:
Received 13 November 2008
Received in revised form 27 February 2009
Available online 13 May 2009
Communicated by A.V. Geramita

MSC:
13F05
13B05
13A15
13D05
13B25

ABSTRACT

This paper deals with well-known extensions of the Prüfer domain concept to arbitrary commutative rings. We investigate the transfer of these notions in trivial ring extensions (also called idealizations) of commutative rings by modules and then generate original families of rings with zero-divisors subject to various Prüfer conditions. The new examples give further evidence for the validity of the Bazzoni–Glaz conjecture on the weak global dimension of Gaussian rings. Moreover, trivial ring extensions allow us to widen the scope of validity of Kaplansky–Tsang conjecture on the content ideal of Gaussian polynomials.

1. Introduction

All rings considered in this paper are commutative with identity elements and all modules are unital. In 1932, Prüfer introduced and studied integral domains in which every non-zero finitely generated ideal is invertible [24]. In 1936, Krull [20] named these rings after H. Prüfer and stated equivalent conditions for a ring to be a Prüfer domain. Since then, "Prüfer domains have assumed a central role in the development of multiplicative ideal theory through numeral equivalent forms. These touched on many areas of commutative algebra, e.g., valuation theory, arithmetic relations on the set of ideals, $*$-operations, and polynomial rings; in addition to several homological characterizations" (Gilmer [9]).

The extension of this concept to rings with zero-divisors gives rise to five classes of Prüfer-like rings featuring some homological aspects (Bazzoni–Glaz [2] and Glaz [10]). At this point, we make the following definition:

Definition 1.1. Let R be a commutative ring.

(1) R is called *semi-hereditary* if every finitely generated ideal of R is projective [5].
(2) R is said to have *weak global dimension* ≤ 1 ($\mathrm{w.\,gl.\,dim}(R) \leq 1$) if every finitely generated ideal of R is flat [11,12].
(3) R is called an *arithmetical ring* if the lattice formed by its ideals is distributive [6].
(4) R is called a *Gaussian ring* if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$ [26].
(5) R is called a *Prüfer ring* if every finitely generated regular ideal of R is invertible [4,15].

In the domain context, all these forms coincide with the definition of a Prüfer domain. Glaz [10] provides examples which show that all these notions are distinct in the context of arbitrary rings. The following diagram of implications summarizes the relations between them [2,3,12,10,21,22,26]:

$$\text{Semi-hereditary} \Rightarrow \text{weak global dimension} \leq 1 \Rightarrow \text{Arithmetical} \Rightarrow \text{Gaussian} \Rightarrow \text{Prüfer}$$

In this paper, we investigate the transfer of the five Prüfer conditions in trivial ring extensions, which are defined below.

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0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2009.04.011

Definition 1.2. Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$.

For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R; ideals of R need not be of this form [19, Example 2.5]. However, prime (resp., maximal) ideals of R have the form $p \propto E$, where p is a prime (resp., maximal) ideal of A [17, Theorem 25.1(3)]. Suitable background on commutative trivial ring extensions is [11,17].

It is notable that original examples, for each one of the above classes, are rare in the literature. This paper investigates the transfer of the aforementioned Prüfer conditions to trivial ring extensions. Our results generate new examples which enrich the current literature with new families of Prüfer-like rings with zero-divisors. In particular, we obtain further evidence for the validity of Bazzoni–Glaz conjecture sustaining that "the weak global dimension of a Gaussian ring is 0, 1, or ∞" [3]. Moreover, trivial ring extensions offer the possibility to widen the scope of validity of the content conjecture of Kaplansky and Tsang which was extended to investigate rings where "every Gaussian polynomial has locally principal content ideal" [1,2,14,16,21,22,26]. Notice that both conjectures share the common context of rings with zero-divisors. This very fact lies behind our motivation for studying the Gaussian condition and related concepts in trivial ring extensions.

Section 2 deals with trivial ring extensions of the form $R := A \propto B$, where $A \subseteq B$ is an extension of integral domains. The main result asserts that "R is Gaussian (resp., Arithmetical) if and only if A is Prüfer with $K \subseteq B$ (resp., $K = B$)." This generates new examples of non-arithmetical Gaussian rings as well as arithmetical rings with weak global dimension strictly greater than one. Recall that classical examples of non-semi-hereditary arithmetical rings stem from Jensen's 1966 result [18] as non-reduced principal rings, e.g., $\mathbb{Z}/n^2\mathbb{Z}$ for any integer $n \geq 2$. In this respect, we provide a new family of examples of non-finite conductor arithmetical rings, hence quite far from being principal. We also establish a result on the weak global dimension of these constructions which happens to corroborate the Bazzoni–Glaz conjecture (cited above).

In their recent paper devoted to Gaussian properties, Bazzoni and Glaz have proved that a Prüfer ring satisfies any of the other four Prüfer conditions if and only if its total ring of quotients satisfies that same condition [3, Theorems 3.3 & 3.6 & 3.7 & 3.12]. This fact narrows the scope of study to the class of total rings of quotients. Section 3 investigates Prüfer conditions in a special class of total rings of quotients; namely, those arising as trivial ring extensions of local rings by vector spaces over the residue fields. The main result establishes that if (A, M) is a non-trivial local ring and E a nonzero $\frac{A}{M}$-vector space, then $R := A \propto E$ is a non-arithmetical total ring of quotients. Moreover, R is a Gaussian ring if and only if A is a Gaussian ring. This enables us to build new examples of non-arithmetical Gaussian total rings of quotients or non-Gaussian total rings of quotients (which are necessarily Prüfer). Furthermore, the weak global dimension of these constructions turns out to be infinite when M admits a minimal generating set.

A problem initially associated with Kaplansky and his student Tsang [1,2,14,22,26] and also termed as Tsang–Glaz–Vasconcelos conjecture in [16] sustained that "every nonzero Gaussian polynomial over a domain has an invertible (or, equivalently, locally principal) content ideal." It is well-known that a polynomial over any ring is Gaussian if its content ideal is locally principal. The converse is precisely the object of Kaplansky–Tsang–Glaz–Vasconcelos conjecture extended to those rings where "every Gaussian polynomial has locally principal content ideal. The objective of Section 4 is to validate this conjecture in a large family of rings distinct from the three classes of arithmetical rings, of locally domains, and of locally approximately Gorenstein rings, where the conjecture holds so far. This is made possible by the main result which states that a trivial ring extension of a domain by its quotient field satisfies the condition that "every Gaussian polynomial has locally principal content ideal." We end up with a conjecture that equates the latter condition with the local irreducibility of the zero ideal. This would offer an optimal solution to the Kaplansky–Tsang–Glaz–Vasconcelos conjecture that recovers all previous results. The section closes with a discussion –backed with examples– which attempts to rationalize this statement.

2. Extensions of domains

This section explores trivial ring extensions of the form $R := A \propto B$, where $A \subseteq B$ is an extension of integral domains. Notice in this context that $(a, b) \in R$ is regular if and only if $a \neq 0$. The main result (Theorem 2.1) examines the transfer of Prüfer conditions to R and hence generates new examples of non-arithmetical Gaussian rings and of arithmetical rings with weak global dimension $\gneqq 1$.

In 1969, Osofsky proved that the weak global dimension of an arithmetical ring is either ≤ 1 or infinite [23]. In 2005, Glaz proved Osofsky's result in the class of coherent Gaussian rings [12, Theorem 3.3]. Recently, Bazzoni and Glaz conjectured that "the weak global dimension of a Gaussian ring is 0, 1, or ∞" [3]. Theorem 2.1 validates this conjecture for the class of all Gaussian rings emanating from these constructions. Moreover, Example 2.7 widens its scope of validity beyond coherent Gaussian rings.

Theorem 2.1. *Let $A \subseteq B$ be an extension of domains and $K := \mathrm{qf}(A)$. Let $R := A \propto B$ be the trivial ring extension of A by B. Then:*

(1) *R is Gaussian if and only if R is Prüfer if and only if A is Prüfer with $K \subseteq B$.*
(2) *R is arithmetical if and only if A is Prüfer with $K = B$.*
(3) $\mathrm{w.\,gl.\,dim}(R) = \infty$.