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The inversion formula for automorphisms of the Weyl algebras and polynomial algebras

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Abstract

Let A_n be the nth *Weyl algebra* and P_m be a polynomial algebra in m variables over a field K of characteristic zero. The following characterization of the algebras $\{A_n \otimes P_m\}$ is proved: *an algebra A admits a finite set* $\delta_1, \dots, \delta_s$ *of commuting locally nilpotent derivations with generic kernels and* $\cap_{i=1}^{s} \ker(\delta_i) = K$ *iff* $A \simeq A_n \otimes P_m$ *for some n and m with* $2n + m = s$, *and vice versa. The inversion formula* for automorphisms of the algebra $A_n \otimes P_m$ (and for $\widehat{P}_m := K[[x_1, \dots, x_m]]$) has been found (giving a *new* inversion formula even for polynomials). Recall that (see [H. Bass, E.H. Connell, D. Wright, The Jacobian Conjecture: Reduction of degree and formal expansion of the inverse, Bull. Amer. Math. Soc. (New Series) 7 (1982) 287–330]) *given* $\sigma \in \mathrm{Aut}_K(P_m)$, *then* $\deg \sigma^{-1} \leq (\deg \sigma)^{m-1}$ (the proof is *algebro-geometric*). We extend this result (using [non-holonomic] $\mathcal{D}$-*modules*): *given* $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$, *then* $\deg \sigma^{-1} \leq (\deg \sigma)^{2n+m-1}$. Any automorphism $\sigma \in \mathrm{Aut}_K(P_m)$ is determined by its *face* polynomials [J.H. McKay, S.S.-S. Wang, On the inversion formula for two polynomials in two variables, J. Pure Appl. Algebra 52 (1988) 102–119], a *similar* result is proved for $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$.

One can amalgamate two old open problems (**the Jacobian Conjecture** and **the Dixmier Problem**, see [J. Dixmier, Sur les algèbres de Weyl, Bull. Soc. Math. France 96 (1968) 209–242. [6]] problem 1) into a single question, (**JD**): *is a K-algebra endomorphism* $\sigma : A_n \otimes P_m \to A_n \otimes P_m$ *an algebra automorphism provided* $\sigma(P_m) \subseteq P_m$ *and* $\det(\frac{\partial \sigma(x_i)}{\partial x_j}) \in K^* := K \setminus \{0\}$? ($P_m = K[x_1, \dots, x_m]$). It follows immediately from the inversion formula that *this question has an affirmative answer iff both conjectures have* (see below) [*iff one of the conjectures has a positive answer* (as follows from the recent papers [Y. Tsuchimoto, Endomorphisms of Weyl algebra and p-curvatures, Osaka J. Math. 42(2) (2005) 435–452. [10]] and [A. Belov-Kanel, M. Kontsevich, The Jacobian conjecture is stably equivalent to the Dixmier Conjecture. ArXiv:math.RA/0512171. [5]])].

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1. Introduction

The following notation will remain **fixed** throughout the paper (if not stated otherwise): K is a field of characteristic zero (not necessarily algebraically closed), module means a *left* module, $A_n = \oplus_{\alpha \in \mathbb{N}^{2n}} K x^\alpha$ is the nth *Weyl algebra* over K (the commutator $[x_{n+i}, x_j] = \delta_{ij}$, $1 \leq i, j \leq n$, where δ_{ij} is the *Kronecker* delta), $P_m = \oplus_{\alpha \in \mathbb{N}^m} K x^\alpha$ is

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a *polynomial* algebra over K (in m variables $x_{2n+1}, \dots, x_{2n+m}$), $A := A_n \otimes P_m = \oplus_{\alpha \in \mathbb{N}^s} Kx^\alpha$, $x^\alpha := x_1^{\alpha_1} \cdots x_s^{\alpha_s}$, $s := 2n + m$, is the Weyl algebra with polynomial coefficients where $x_1, \dots, x_s$ are the canonical generators for A (see below). Any K-algebra automorphism $\sigma \in \mathrm{Aut}_K(A)$ is uniquely determined by the elements $x_i' := \sigma(x_i) = \sum_{\alpha \in \mathbb{N}^s} \lambda_\alpha x^\alpha$, $i = 1, \dots, s$, $\lambda_\alpha \in K$, and so is its inverse, $\sigma^{-1}(x_i) = \sum_{\alpha \in \mathbb{N}^s} \lambda_\alpha' x^\alpha$, $i = 1, \dots, s$.

What is the inversion formula for $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$**?** A natural (shortest) answer to this question is a formula for the coefficients $\lambda_\alpha' = \lambda_\alpha'(\lambda_\beta)$ *like the inversion formula (Cramer's formula) in the linear polynomial case*: given $x' = Ax$ where $A = (a_{ij}) \in \mathrm{GL}_m(K)$ (i.e. $x_i' = \sum_{j=1}^m a_{ij} x_j$ where $a_{ij} = \frac{\partial x_i'}{\partial x_j} \in K$) then

$$x = A^{-1}x' = \left(\frac{\partial x_i'}{\partial x_j}\right)^{-1} x' = (\det A)^{-1}(\Delta_{ij})x'$$

where Δ_{ij} are complementary minors for the matrix $(\frac{\partial x_i'}{\partial x_j})$; these are linear combinations of products of partial derivatives $\frac{\partial x_i'}{\partial x_j}$. So, the inversion formula, in the general situation, is a formula, $\lambda_\alpha' = \lambda_\alpha'(\lambda_\beta)$, where *only* additions and multiplications are allowed of 'partial derivatives' of the elements x' (taking partial derivatives 'corresponds' to the operation of taking coefficients of x'). So, *the* inversion formula is the most *economical* formula (the point I want to make is that $x = \frac{1}{2}x'$ is the inversion formula for the equation $x' = 2x$ but $x = \frac{1}{2}(x' + 2\int_0^1 f(t)\,\mathrm{d}t + 2\dim_K \mathrm{Ext}_B^i(M, N)) - \int_0^1 f(t)\,\mathrm{d}t - \dim_K \mathrm{Ext}_B^i(M, N)$ is 'not').

Theorem 2.4 gives *the inversion formula* for an automorphism $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. Theorem 4.3 gives a *similar formula* for an automorphism $\sigma \in \mathrm{Aut}_K(K[[x_1, \dots, x_m]])$. For *another* inversion formula for $\sigma \in \mathrm{Aut}_K(P_m)$ see [3,1].

The degree of σ^{-1} **where** $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. We extend the following result which according to the comment made on p. 292, [3]: 'was "well-known" to the classical geometers' and 'was communicated to us [H. Bass, E.H. Connell, D. Wright] by Ofer Gabber .... He attributes it to an unrecalled colloquium lecture at Harward'.

Theorem 1.1 (*[3,9]*). *Given* $\sigma \in \mathrm{Aut}_K(P_m)$, *then* $\deg \sigma^{-1} \le (\deg \sigma)^{m-1}$.

The proof of this theorem is *algebro-geometric* (see [2] for a generalization of this result for certain varieties). We extend this result (see Section 3).

Theorem 1.2. *Given* $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. *Then* $\deg \sigma^{-1} \le (\deg \sigma)^{2n+m-1}$.

Non-holonomic $\mathcal{D}$-modules are used in the proof (it looks like this is one of the first instances where *non-holonomic* $\mathcal{D}$-modules are of real use).

The algebras $\{A_n \otimes P_m\}$ **as a class**. Theorem 5.3 gives a characterization of the algebras $\{A_n \otimes P_m\}$ as a class via commuting sets of locally nilpotent derivations: *an algebra A admits a finite set* $\delta_1, \dots, \delta_s$ *of commuting locally nilpotent derivations with generic kernels and* $\cap_{i=1}^s \ker(\delta_i) = K$ *iff* $A \simeq A_n \otimes P_m$ *for some n and m with* $2n+m = s$, *and vice versa* (the kernels $\ker(\delta_i)$ are *generic* if the intersections $\{\cap_{i=1}^s \ker(\delta_i), \cap_{i \ne j} \ker(\delta_i) \mid j = 1, \dots, s\}$ are *distinct*).

Left and right faces of an automorphism $\sigma \in \mathrm{Aut}_K(A_n \otimes P_m)$. Let $P_m = K[X_1, \dots, X_m]$ be a polynomial algebra. For each $i = 1, \dots, m$, the *algebra epimorphism* $f_i : P_m \to P_m/(X_i)$, $p \mapsto p + (X_i)$, is called the *face homomorphism*. McKay and Wang [8] proved: *given* $\sigma, \tau \in \mathrm{Aut}_K(P_m)$ *such that* $f_i\sigma = f_i\tau$, $i = 1, \dots, m$, *then* $\sigma = \tau$. So, an automorphism $\sigma \in \mathrm{Aut}_K(P_m)$ is completely determined by its *faces* $\{f_i\sigma \mid i = 1, \dots, m\}$ or, equivalently, by its *face polynomials* $\{f_i\sigma(X_j) \mid i, j = 1, \dots, m\}$ since each map $f_i\sigma$ is an *algebra homomorphism*.

For the algebra $A := A_n \otimes P_m = K\langle x_1, \dots, x_s\rangle$, $s := 2n + m$, (where $x_1, \dots, x_s$ are the canonical generators) we have *left faces* $l_i : A \to A/Ax_i$, $a \mapsto a + Ax_i$, and *right faces* $r_i : A \to A/x_iA$, $a \mapsto a + x_iA$, $i = 1, \dots, s$. These are homomorphisms of *left* and *right* A-modules rather than homomorphisms of algebras (if $x_i \in P_m$ then $l_i = r_i$ is an *algebra homomorphism*).

Theorem 6.1 states that: *given* $\sigma, \tau \in \mathrm{Aut}_K(A)$ *such that* $r_i\sigma = r_i\tau$, $i = 1, \dots, s$ *then* $\sigma = \tau$ (*similarly,* $l_i\sigma = l_i\tau$, $i = 1, \dots, s$, *imply* $\sigma = \tau$).

2. The inversion formula

In this section, the inversion formula (Theorem 2.4) is given.