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The group of endotrivial modules in the normal case

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Abstract

The group of endotrivial modules has recently been determined for a finite group having a normal Sylow p-subgroup. In this paper, we give and compare three different presentations of a torsion-free subgroup of maximal rank of the group of endotrivial modules. Finally, we illustrate the constructions in an example.

MSC: 20C20

1. Introduction

Endotrivial modules play an important role in the modular representation theory of finite groups, and this may explain why many group theorists have been studying them intensively, since the late seventies. The classification of endotrivial modules of a finite p-group was recently achieved in [11]. Thereafter, in a joint work, Jon Carlson, Daniel Nakano and the author tackled the question of the classification of endotrivial modules for an arbitrary finite group, and they were able to give an almost complete classification in the case of a finite group of Lie type in its defining characteristic (cf. [8]). The general case is still an open question, in the sense that no presentation by generators and relations of the group of endotrivial modules is yet known. However, the obstacles have been overcome in the case of a finite group having a normal Sylow p-subgroup. The results are presented in [8, Theorem 3.4], where the authors show that in this case, the group of endotrivial modules is generated by the classes of the indecomposable endotrivial modules that are extended from the Sylow p-subgroup. Then, by means of cohomological tools, they construct a minimal set of generators for the group of endotrivial modules.

The primary aim of these notes is to give an alternative construction of a torsion-free subgroup of maximal rank of the group of endotrivial modules of a finite group having a normal Sylow p-subgroup, that does not appeal to any cohomological knowledge. The method refers to a theorem proven by Dade, and it is presented in Section 3; after that basic facts about endotrivial modules are recapitulated in Section 2. In Section 4, we review the modules and the techniques used in [7,8], and we compare with the approach presented in the previous section. Finally, in Section 5, we work out thoroughly an “odd extraspecial” example.

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2. Preliminaries

Throughout these notes, we let k be an algebraically closed field of prime characteristic p. If G is a finite group, we write $\mathbf{mod}(kG)$ for the category of finitely generated kG-modules and $\mathbf{stmod}(kG)$ for the stable module category. That is, the objects of $\mathbf{stmod}(kG)$ are the same as those of $\mathbf{mod}(kG)$, and the morphisms are equivalence classes of morphisms. Namely, two morphisms are equivalent if their difference factors through a projective module (cf. [6, Section 5]). In addition, we write k for the one-dimensional trivial kG-module, and if M is a finitely generated kG-module, then $\mathrm{End}_k M = \mathrm{Hom}_k(M, M)$ denotes the kG-module that is the k-algebra of k-linear endomorphisms of M. Recall that, for two kG-modules M and N, there is an isomorphism of kG-modules $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$, where $M^* = \mathrm{Hom}_k(M, k)$ is the k-linear dual of M, and the tensor product "$\otimes$" is the tensor product over the field k.

Definition 2.1. Let G be a finite group. A finitely generated kG-module M is *endotrivial* provided that $\mathrm{End}_k M \cong k$ in $\mathbf{stmod}(kG)$, or equivalently, $\mathrm{End}_k M \cong k \oplus (\mathrm{proj})$ in $\mathbf{mod}(kG)$, for some projective kG-module (proj).

We say that two endotrivial kG-modules are *equivalent* if they are isomorphic in $\mathbf{stmod}(kG)$. The set $T(G)$ of isomorphism classes in $\mathbf{stmod}(kG)$ of endotrivial kG-modules is an abelian group, called the *group of endotrivial modules*. The composition law is defined by $[M] + [N] = [M \otimes N]$.

In particular, in $T(G)$, we have $0 = [k]$ and $-[M] = [M^*]$.

Endotrivial kG-modules were defined by Dade (cf. [12]), in 1978, for finite p-groups, as a particular case of the capped endo-permutation kG-modules. A *capped endo-permutation kG-module*, for a finite p-group G, is a finitely generated kG-module whose endomorphism algebra is a permutation module having a trivial direct summand. Modulo a suitable equivalence relation, they form a finitely generated abelian group $D(G)$, and the group $T(G)$ identifies with a subgroup of $D(G)$.

Also, for a subgroup H of G, the restriction map $\mathrm{Res}^G_H : \mathbf{mod}(kG) \to \mathbf{mod}(kH)$ (also denoted by "$\cdot\downarrow^G_H$") induces a group homomorphism $\mathrm{Res}^G_H : T(G) \to T(H)$.

Non-trivial examples of endotrivial modules are the syzygies of the trivial module, whereas, in the case of a finite p-group, most of the relative syzygies are capped endo-permutation modules (not endotrivial in general; cf. [1]). Let us recall their definitions.

Definition 2.2. If X is a finite G-set, then $\Omega^1_X(k)$ is the *relative* (*to* X) *syzygy of* k, that is, the kernel of the augmentation map $kX \to k$.

If $X = G$, then we define the *syzygy* $\Omega^n_G(k)$ *of* k, for each $n \in \mathbb{Z}$, as follows. If $n \geq 1$, we let $\Omega^n_G(k)$ be the kernel of the $(n-1)$-st differential in a minimal projective resolution of k.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+1}"] & X_n \arrow[rr, "\partial_n"] \arrow[dr, two heads] & & X_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_2"] & X_1 \arrow[rr, "\partial_1"] \arrow[dr, two heads] & & X_0 \arrow[r, two heads, "\partial_0"] & k \\
& & \Omega^n_G(k) \arrow[ur, hook] & & & & \Omega^1_G(k) \arrow[ur, hook] & &
\end{tikzcd}$$

If $n \leq -1$, we let $\Omega^n_G(k) = \mathrm{Hom}_k(\Omega^{-n}_G(k), k)$, and we set $\Omega^0_G(k) = k$.

Let G be a p-group and suppose that $\Omega^1_X(k)$ is a capped endo-permutation kG-module. Then we let Ω_X denote its class in $D(G)$, or $T(G)$ in the case where it is endotrivial. In particular, for any finite group G, and for any integer n, the syzygies $\Omega^n_G(k)$ are indecomposable endotrivial modules and we have $[\Omega^n_G(k)] = n\Omega_G$ in $T(G)$. We refer the reader to Section 4 in [6] for more properties of the syzygies, and to Sections 2–5 in [4] for those of the relative syzygies.

Elementary abelian p-subgroups play an important role in the analysis of $T(G)$. In particular, we will need the following group theoretical notions for our purposes.

Definition 2.3. Let G be a finite group and p be a prime.

(1) The *p-rank of* G is the largest integer r such that G has an elementary abelian p-subgroup of rank r.
(2) We write $\mathcal{E}_{\geq 2}(G)$ for the poset of G-conjugacy classes of elementary abelian p-subgroups of G of p-rank at least 2.