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Valuations dominating regular local rings and proximity relations[☆]

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Abstract

We study zero-dimensional valuations dominating a regular local ring of dimension $n \geq 2$. For this we introduce the proximity matrix and the multiplicity sequence (extending classical definitions of the case $n = 2$) that are associated with the sequence of the successive quadratic transforms of the ring along the valuation. We describe the precise relations between these invariants and study their properties.

MSC: 13F30; 13H05

1. Introduction

The proximity relations (codified as proximity matrix, Enriques graph, dual graph, ...) play a central role in the study of plane curve singularities, complete ideals in two-dimensional regular rings, valuations dominating two-dimensional regular rings, ... (See [4,5,9,11], ...) They are invariants of the geometry of the plane curve resolution, of the base points of complete ideals, of the successive quadratic transform along the valuation, ... The problem in greater dimension is much more complicated than for the two-dimensional case.

The main purpose of this paper is to study the proximity relations for valuations centered in regular local rings of any dimension. More precisely, let R be a regular noetherian local ring of dimension $n \geq 2$ and let V be a valuation ring of the quotient field of R such that V dominates R (i. e. $R \subset V$ and $M(V) \cap R = M(R)$, where $M(R)$ and $M(V)$ are the maximal ideals of R and V respectively). Let us denote by v the valuation associated with V and assume that v is a zero-dimensional valuation, i.e. the extension of residual fields $R/M(R) \subset V/M(V)$ is an algebraic extension. Associated with the pair (R, V), we have the sequence of regular noetherian local rings $(R_i) \equiv R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_i \subset \cdots \subset V$ such that R_i is the quadratic transform of R_{i-1} along V, $i \geq 1$. Since v is zero-dimensional, then all the rings R_i have the same dimension n.

A first question is to decide when V is determined by the sequence (R_i), or equivalently, when $V = \cup_{i=0}^{\infty} R_i$. In the case where $n = 2$, it is well known that $V = \cup_{i=0}^{\infty} R_i$ (see [1]). For $n \geq 3$ there are pairs (R, V) such that $V \neq \cup_{i=0}^{\infty} R_i$,

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in fact in this case there exist infinitely many valuation rings V dominating $\cup_{i=0}^{\infty} R_i$. (See Proposition 5.) However, Shannon [10] characterizes the sequences (R_i) for which $V = \cup_{i=0}^{\infty} R_i$, when v is a non-discrete valuation of real rank one. In [7], this characterization is completed without any assumption on the real rank of v.

With this background, a related second question is: what information about v can be recovered from the sequence (R_i) and viceversa?

To answer this we consider the proximity matrix obtained directly from the sequence (R_i) and the multiplicity sequence corresponding to v.

Following [9], R_j is said to be proximate to R_i, $i < j$, if R_j is contained in the valuation ring of the usual order valuation Ord_{R_i} of R_i. These proximity relations are collected into the proximity matrix $P = (p_{ij})$, where $p_{ii} = 1$, $p_{ij} = -1$ if $i < j$ and R_j is proximate to R_i and $p_{ij} = 0$ otherwise, $i, j \geq 0$.

The multiplicity sequence is the sequence $\{n_i\}_{i \geq 0}$ given by $n_i = \min\{v(y);\ y \in M(R_i) - \{0\}\}$, where $M(R_i)$ is the maximal ideal of R_i, $i \geq 0$.

As we have said, for $n = 2$ these and others invariants have been studied by several authors. See for example [4,5,9,11], ...

In general, the proximity matrix is determined by the multiplicity sequence (Corollary 21), but the converse is not true, even for $n = 2$ (Remark 22). However, the proximity matrix and the multiplicity sequence are equivalent data associated with the pair (R, V), when v has real rank one and $V = \cup_{i=0}^{\infty} R_i$ (Shannon's case), see [9] for the case $n = 2$ and [8] for any n.

Now, the second question can be raised as follows: what information about the multiplicity sequence is determined by the proximity matrix?

Roughly speaking one can say that the proximity matrix and the multiplicity sequence are equivalent data after deleting some finite information in the proximity matrix and taking a convenient projection of the multiplicity sequence. Namely, for each proximity matrix there exists a non-negative integer $N_0(P)$ such that for each $i \geq N_0(P)$ there exists a non-negative integer $h(i)$ with $p_{ij} = 0$ for $j > i + h(i)$, and $p_{N_0(P)-1,j} \neq 0$ for $j \geq N_0(P) - 1$. (See Lemma 12 and Definition 13.)

On the other hand, let us denote by Γ the value group of v, that we identify with a subgroup of $\mathbb{R}^r$ lexicographically ordered, where $\mathbb{R}$ is the set of real numbers and r is the real rank of v. Then there exists a non-negative integer h with $1 \leq h \leq r$ and a non-negative integer $N_0(v)$ such that $pr_j(n_i) = 0$ and $pr_h(n_i) > 0$ for $i \geq N_0(v)$ and $1 \leq j < h$. Here $pr_j : \mathbb{R}^r \longrightarrow \mathbb{R}$ is the usual jth projection, $1 \leq j \leq r$. (See Lemma 24.)

In Proposition 27 we get that $N_0(v) \leq N_0(P)$ and in Theorem 29 we show that the matrix ${}^{N_0(P)}P$ and the sequence $\{(pr_h(n_{N_0(P)})/pr_h(n_i))\}_{i \geq N_0(P)}$ are equivalent data associated with the pair (R, V), where ${}^{N_0(P)}P$ is the matrix obtained from P by deleting the first $N_0(P)$ rows and columns, i.e. ${}^{N_0(P)}P$ is the proximity matrix associated with the sequence $R_{N_0(P)} \subset R_{N_0(P)+1} \subset R_{N_0(P)+2} \subset \cdots \subset R_i \subset \ldots$.

Finally and in relation with the first question raised, we give sufficient conditions on the proximity matrix and the multiplicity sequence to get $V = \cup_{i=0}^{\infty} R_i$. (See Theorem 16 and Proposition 23.)

2. Quadratic transformations

Most of the concepts and notations in this paper are the same as in [1–3,8,10,12]. Several of them are recapitulated in this section.

All the rings considered are commutative and with unit element.

For a noetherian local ring R, we denote by $M(R)$ the maximal ideal of R and by $\dim(R)$ the Krull dimension of R. Also, for each non-zero principal ideal J of R we denote by $\mathrm{Ord}_R(J)$ the usual multiplicity of J, i.e. the non-negative integer d such that $J \subset (M(R))^d$ and $J \not\subset (M(R))^{d+1}$.

If $\mathfrak{a}$ is a non-zero ideal of R, a *monoidal transform of R with center* $\mathfrak{a}$ is a ring $R_1 = (R[\mathfrak{a}z^{-1}])_{\mathfrak{q}}$, where z is a non-zero element of $\mathfrak{a}$ and $\mathfrak{q}$ is a prime ideal of $R[\mathfrak{a}z^{-1}]$ such that $M(R)R[\mathfrak{a}z^{-1}] \subset \mathfrak{q}$. In the case where $\mathfrak{a} = M(R)$, R_1 is called a *quadratic transform* of R and for any base $(y_1, \ldots, y_n)$ of $M(R)$ we can write $R_1 = (R[\frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1}])_Q$, where Q is a prime ideal of $R[\frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1}]$ with $M(R) \subset Q$.

Let J be a non-zero principal ideal of R and let R_1 be a quadratic transform of R. The *strict transform* of (R, J) in R_1 is the pair (R_1, J_1), where J_1 is the ideal such that $J_1(M(R))^m R_1 = J R_1$ and $m = \mathrm{Ord}_R(J)$.