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Representations and the Jacobian Conjecture

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Abstract

In this paper we apply the representation theory of the Lie algebra $sl_2(\mathbb{C})$ to the problem of describing Hessian nilpotent polynomials, which are important in the theory of the Jacobian Conjecture. In the two variable case we describe them as the maximal and minimal weight vectors of the irreducible representations of $sl_2(\mathbb{C})$. For the first time this gives a characterization of the Hessian nilpotent polynomials in terms of linear differential operators.

MSC: 14R15; 22E27

1. Introduction

A homogeneous polynomial $f(X_1, X_2, \ldots, X_n)$ whose Hessian matrix is nilpotent is called *Hessian nilpotent*. Hessian nilpotent polynomials are important in the theory of the Jacobian Conjecture.

The techniques of invariant theory and representation theory so far have not been applied to the Jacobian Conjecture. In this paper we explore the idea that this could be done by describing classes of Hessian nilpotent polynomials.

We formulate the two dimensional case of the problem entirely in terms of the representation theory of the Lie algebra $sl_2(\mathbb{C})$ and prove, in particular, that the Hessian nilpotent polynomials are the maximal and minimal weight vectors of the irreducible representations of $sl_2(\mathbb{C})$.

A quote from the paper [6] seems appropriate:

> *Bei der Wichtighkeit, welche die Hesse'che Covarianten eines Form fur diesen hat, haben wir die Untersuchung wieder aufgennomen...*

The Jacobian Conjecture states that a polynomial map

$$f : \mathbb{C}^n \to \mathbb{C}^n$$

has a polynomial inverse if and only if the Jacobian Condition is satisfied:

$$\det(J(f)) \in \mathbb{C}^*$$

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where $J(f)$ is the Jacobian matrix of f. If f is invertible, then the Chain Rule implies that $\det J(f) \in C^*$, so to prove the Conjecture one must prove that if f satisfies the Jacobian Condition then f has a polynomial inverse.

In their basic paper on the Jacobian Conjecture [1] Bass, Connell, and Wright proved that the Conjecture follows if it can be proved to hold for polynomial mappings of the form

$$f(X_k) = X_k + H_k, \quad 1 \le k \le n$$

where $H_1, H_2, \ldots, H_n$ are homogeneous polynomials of degree $d \ge 2$.

They also proved that the Jacobian Condition for such maps f is equivalent to the condition that

$$J(H)^n = 0$$

that is, that the Jacobian matrix of H is nilpotent.

A remarkable theorem of de Bondt and van den Essen [3] asserts that the Jacobian Conjecture follows if it holds for homogeneous polynomial maps with symmetric Jacobian matrix. An application of the Poincaré Lemma shows that the Jacobian Conjecture holds if it holds for all *gradient mappings*

$$f(X_k) = X_k + \partial F/\partial X_k, \quad 1 \le k \le n$$

where F is a homogeneous polynomial. The Jacobian matrix of a gradient map is the identity plus the Hessian matrix

$$(\partial^2 F/\partial X_i \partial X_j)$$

so finally the Jacobian Conjecture follows if it holds for all gradient mappings of homogeneous polynomials of degree two or more whose Hessian matrices are nilpotent.

To prove the Jacobian Conjecture, it is sufficient to invert the gradient map of any Hessian nilpotent polynomial.

In Section 2 below we study polynomials which are *induced* in the obvious sense from polynomials in fewer variables. We study this relation, and prove (Theorem 2.1) that it always gives rise to Hessian nilpotent polynomials.

In Section 3 we give a careful treatment of the two variable case, formulating all our results in terms of the representation theory of the Lie algebra $sl_2(\mathbb{C})$. Our aim is to make it as clear as possible how the results in this case can be generalized to higher dimensions. We show that the Hessian nilpotent polynomials in this case are the maximal and minimal weight vectors for the irreducible representations of $sl_2(\mathbb{C})$.

In Section 4 we pose and discuss several questions which we think are of interest.

This results of this paper were strongly influenced by the work of A. van den Essen and M. de Bondt, and we thank them heartily for their support and encouragement.

2. Induced polynomials

In this section we give a precise definition of polynomials which are induced from polynomials in fewer variables, and prove that they are always Hessian nilpotent. A somewhat more general definition could be made, but we do not see the need for it. A version of this construction was given by Zhao in [10].

Let n be a positive integer, and let $n = n_1 n_2$ be a factorization of n, with $n_1 > 1$ and $n_2 \ge 1$. Let $f(X)$ be a homogeneous polynomial in n variables, and let $g(T)$ be a homogeneous polynomial in n_2 variables. Let

$$L_1 = a_1 x_1 + a_2 x_2 + \cdots + a_{n_1} x_{n_1}$$
$$\cdots$$
$$L_{n_2} = a_1 x_{n-n_1+1} + \cdots + a_{n_1} x_n$$

where the coefficient vector $(a_1, \ldots, a_{n_1})$ is isotropic (the dot product of the vector with itself is zero). If

$$f(x_1, \ldots, x_n) = g(L_1, \ldots, L_{n_2})$$

then f is *induced* from g.

If f is a homogeneous polynomial, denote the Hessian of f by $h(f)$.

Theorem 2.1. *If f is induced from g as above, then $h(f)^2 = 0$, and in particular f is Hessian nilpotent.*