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Operations in Milnor K-theory

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ABSTRACT

We show that operations in Milnor K-theory mod p of a field are spanned by divided power operations. After giving an explicit formula for divided power operations and extending them to some new cases, we determine for all fields k_0 and all prime numbers p, all the operations $K_i^{\mathrm{M}}/p \to K_j^{\mathrm{M}}/p$ commuting with field extensions over the base field k_0. Moreover, the integral case is discussed and we determine the operations $K_i^{\mathrm{M}}/p \to K_j^{\mathrm{M}}/p$ for smooth schemes over a field.

0. Introduction

Let k_0 be a field and p be a prime number different from the characteristic of k_0. In [1], Voevodsky constructs Steenrod operations on the motivic cohomology $H^{*,*}(X, \mathbf{Z}/p)$ of a general scheme over k_0. However, when p is odd or when $p = 2$ and -1 is a square in $k_0^\times$, such operations vanish on the motivic cohomology groups $H^{i,i}(\operatorname{Spec} k, \mathbf{Z}/p)$ for $i > 0$ of the spectrum of a field extension k of k_0. Here, we study operations on $H^{i,i}(\operatorname{Spec} k, \mathbf{Z}/p)$ which are defined only for fields.

The same phenomenon happens in étale cohomology, where Steenrod operations, as defined by Epstein in [2], vanish on the étale cohomology $H^i_{et}(\operatorname{Spec} k, \mathbf{Z}/p)$ of a field if p is odd or if $p = 2$ and $\sqrt{-1} \in k$. Under the assumption of the Bloch–Kato conjecture, our operations give secondary operations relatively to Steenrod operations on the étale cohomology of fields.

Given a base field k_0 and a prime number p, an operation on K_i^{M}/p is a function $K_i^{\mathrm{M}}(k)/p \to K_*^{\mathrm{M}}(k)/p$ defined for all fields k/k_0, compatible with extension of fields. In other words, it is a natural transformation from the functor $K_i^{\mathrm{M}}/p : \mathsf{Fields}_{/k_0} \to \mathsf{Sets}$ to the functor $K_*^{\mathrm{M}}/p : \mathsf{Fields}_{/k_0} \to \mathbf{F}_p - \mathsf{Algebras}$. It is important for our purpose that our operations should be functions and not only additive functions, the reason being that additive operations will appear to be trivial in some sense (see Section 3.5). In these notes, we determine all operations $K_i^{\mathrm{M}}/p \to K_*^{\mathrm{M}}/p$ over any field k_0, no matter if $p \neq \operatorname{char} k_0$ or not. This is striking, especially in the case when $i = 1$.

Let n be a non-negative integer and k any field. Let $x = \sum_{r=1}^{l} s_r$ be a sum of l symbols in $K_i^{\mathrm{M}}(k)/p$, the mod p Milnor K-group of k of degree i. We define the nth divided power of x, given as a sum of symbols, by

$$\gamma_n(x) = \sum_{1 \le l_1 < \cdots < l_n \le l} s_{l_1} \cdot \cdots \cdot s_{l_n} \in K_{ni}^{\mathrm{M}}(k)/p.$$

Such a divided power may depend on the way x has been written as a sum of symbols and thus a well-defined map $\gamma_n : K_i^{\mathrm{M}}(k)/p \to K_{ni}^{\mathrm{M}}(k)/p$ may not exist. However, $\gamma_0(x) = 1$ and $\gamma_1(x) = x$ and as such, γ_0 and γ_1 are always well-defined. The axioms for divided powers (see Properties 2.3) formalize the properties of $\frac{x^n}{n!}$ in a $\mathbf{Q}$-algebra, see [3] for some general discussion of a divided power structure on an ideal in a commutative ring. In his paper [4], Kahn shows that the above formula gives well-defined divided powers $\gamma_n : K_{2i}^{\mathrm{M}}(k)/p \to K_{2ni}^{\mathrm{M}}(k)/p$ for p odd and $\gamma_n : K_i^{\mathrm{M}}(k)/2 \to K_{ni}^{\mathrm{M}}(k)/2$ for

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k containing a square root of -1. Kahn's result is based on previous work by Revoy on divided power algebras, [5]. Divided powers are also mentioned in a letter of Rost to Serre, [6]. In this paper, we show that in these cases, divided powers define operations in the above sense and form a basis for all possible operations on mod p Milnor K-theory.

In the remaining case, when -1 is not a square in the base field k_0, divided powers as defined above are not well-defined on mod 2 Milnor K-theory. However, we will define some new, weaker operations, and show that these new operations are all the possible operations on mod 2 Milnor K-theory.

Precisely, we will prove:

Theorem 1 (p Odd). *Let k_0 be any field, p an odd prime number. The algebra of operations $K_i^{\mathrm{M}}(k)/p \to K_*^{\mathrm{M}}(k)/p$ commuting with field extensions over k_0 is*

- *If $i = 0$, the free $K_*^{\mathrm{M}}(k_0)/p$-module of rank p of functions $\mathbf{F}_p \to K_*^{\mathrm{M}}(k_0)/p$.*
- *If $i \geq 1$ is odd, the free $K_*^{\mathrm{M}}(k_0)/p$-module*

$$K_*^{\mathrm{M}}(k_0)/p \cdot \gamma_0 \oplus K_*^{\mathrm{M}}(k_0)/p \cdot \gamma_1.$$

- *If $i \geq 2$ is even, the free $K_*^{\mathrm{M}}(k_0)/p$-module*

$$\bigoplus_{n\geq 0} K_*^{\mathrm{M}}(k_0)/p \cdot \gamma_n.$$

Theorem 2 ($p = 2$). *Let k_0 be any field. The algebra of operations $K_i^{\mathrm{M}}(k)/2 \to K_*^{\mathrm{M}}(k)/2$ commuting with field extensions over k_0 is*

- *If $i = 0$, the free $K_*^{\mathrm{M}}(k_0)/2$-module of rank 2 of functions $\mathbf{F}_2 \to K_*^{\mathrm{M}}(k_0)/2$.*
- *If $i = 1$, the free $K_*^{\mathrm{M}}(k_0)/2$-module of rank 2, generated by γ_0 and γ_1.*
- *If $i \geq 2$, the $K_*^{\mathrm{M}}(k_0)/2$-module*

$$K_*^{\mathrm{M}}(k_0)/2 \cdot \gamma_0 \oplus K_*^{\mathrm{M}}(k_0)/2 \cdot \gamma_1 \oplus \bigoplus_{n\geq 2} \mathrm{Ker}(\tau_i) \cdot \gamma_n,$$

where $\tau_i : K_*^{\mathrm{M}}(k_0)/2 \to K_*^{\mathrm{M}}(k_0)/2$ is the map $x \mapsto \{-1\}^{i-1} \cdot x$.

Actually, the divided powers γ_n are not always defined with the assumptions of Theorem 2. However, if $y_n \in \mathrm{Ker}(\tau_i)$, the map $y_n \cdot \gamma_n$ will be shown to be well-defined. Notice that when -1 is a square in k_0, the map τ_i is the zero map, and hence $\mathrm{Ker}(\tau_i) = K_*^{\mathrm{M}}(k_0)/2$.

Also, the divided powers satisfy the relation $\gamma_m \cdot \gamma_n = \binom{m+n}{n} \gamma_{m+n}$. Together with the algebra structure on $K_*^{\mathrm{M}}(k_0)/p$, this gives the algebra structure of the algebra of operations $K_i^{\mathrm{M}}/p \to K_*^{\mathrm{M}}/p$ over k_0. In fact, divided powers satisfy all the relations mentioned in Properties 2.3 and make Milnor K-theory into a divided power algebra in Revoy's notation [5].

As Nesterenko–Suslin [7] and Totaro [8] have shown, there is an isomorphism $H^{n,n}(\mathrm{Spec}\ k, \mathbf{Z}) \xrightarrow{\simeq} K_n^{\mathrm{M}}(k)$ where $H^{n,n}(\mathrm{Spec}\ k, \mathbf{Z})$ denotes motivic cohomology. This isomorphism, together with Theorem 1, provides operations on the motivic cohomology groups $H^{n,n}(\mathrm{Spec}\ k, \mathbf{Z}/p)$. Also, since the Bloch–Kato conjecture seems to have been proven by Rost and Voevodsky (see [9] and Weibel's paper [10] that patches the overall proof by using operations from integral cohomology to $\mathbf{Z}/p$ cohomology avoiding Lemma 2.2 of [9] which seems to be false as stated), this gives the operations in Galois cohomology of fields, with suitable coefficients.

We also describe some new operations in integral Milnor K-theory over any base field k_0. Under some reasonable hypothesis on an operation $\varphi : K_i^{\mathrm{M}} \to K_*^{\mathrm{M}}$ defined over k_0, we are able to show that φ is in the $K_*^{\mathrm{M}}(k_0)$-span of our weak divided power operations. See Sections 2.5 and 3.4.

Finally, we are able to determine operations $K_i^{\mathrm{M}}/p \to K_j^{\mathrm{M}}/p$ in the more general setup of smooth schemes over a field k. The Milnor K-theory ring of a smooth scheme X over k is defined to be the subring of the Milnor K-theory of the function field $k(X)$ whose elements are unramified along all divisors on X, i.e. which vanish under all residue maps corresponding to codimension-1 points in X. An operation $K_i^{\mathrm{M}}/p \to K_j^{\mathrm{M}}/p$ over the smooth k-scheme X is a function that is functorial with respect to morphisms of X-schemes (see Section 4). Once again, if p is odd and $i \geq 2$ is even, or if $p = 2$ and k contains a square-root of -1, we have

Theorem 3. *Operations $K_i^{\mathrm{M}}/p \to K_*^{\mathrm{M}}/p$ over the smooth k-scheme X are spanned as a $K_*^{\mathrm{M}}(X)/p$-module by the divided power operations.*

Assuming the Bloch–Kato conjecture, we obtain in this way all the operations for the unramified cohomology of smooth schemes over the field k.

The paper is organised as follows. In the first section, we start by recalling some general facts about Milnor K-theory, particularly the existence of residue and specialization maps. In the second section, we give a detailed account on divided power operations and extend the results mentioned in [4] to the case $p = 2, \sqrt{-1} \notin k^\times$. We also describe some weak divided